Long wave approximation for water waves under a Coriolis forcing and the Ostrovsky equation

Benjamin MELINAND*

April 2016

Abstract

This paper is devoted to the study of the long wave approximation for water waves under the influence of the gravity and a Coriolis forcing. We start by deriving a generalization of the Boussinesq equations in 1D (in space) and we rigorously justify them as an asymptotic model of the water waves equations. These new Boussinesq equations are not the classical Boussinesq equations. A new term due to the vorticity and the Coriolis forcing appears that can not be neglected. Then, we study the Boussinesq regime and we derive and fully justify different asymptotic models when the bottom is flat : a linear equation linked to the Klein-Gordon equation admitting the so-called Poincaré waves; the Ostrovsky equation, which is a generalization of the KdV equation in presence of a Coriolis forcing, when the rotation is weak; and finally the KdV equation when the rotation is very weak. Therefore, this work provides the first mathematical justification of the Ostrovsky equation. Finally, we derive a generalization of the Green-Naghdi equations in 1D in space for small topography variations and we show that this model is consistent with the water waves equations.

1 Introduction

We study the motion of an incompressible, inviscid fluid with a constant density ρ and no surface tension under the influence of the gravity $\boldsymbol{g} = -g\boldsymbol{e_z}$ and the rotation of the Earth with a rotation vector $\mathbf{f} = \frac{f}{2}\boldsymbol{e_z}$. We suppose that the seabed and the surface are graphs above the still water level. The horizontal variable is $X = (x, y) \in \mathbb{R}^2$ and $z \in \mathbb{R}$ is the vertical variable. The water occupies the domain $\Omega_t := \{(X, z) \in \mathbb{R}^3 \ , \ -H + b(X) < z < \zeta(t, X)\}$. The velocity in the fluid domain is denoted $\mathbf{U} = (\mathbf{V}, \mathrm{w})^t$ where $\mathbf{V}$ is the horizontal component of $\mathbf{U}$ and w its vertical component. The equations governing such a fluid are the free surface Euler-Coriolis equations[(1)]

$$\begin{cases} \partial_t \mathbf{U} + \left(\mathbf{U} \cdot \nabla_{X,z}\right) \mathbf{U} + \mathbf{f} \times \mathbf{U} = -\dfrac{1}{\rho} \nabla_{X,z} \mathcal{P} - g\boldsymbol{e_z} \text{ in } \Omega_t, \\ \text{div } \mathbf{U} = 0 \text{ in } \Omega_t, \end{cases} \tag{1}$$

with the boundary conditions

*IMB, Université de Bordeaux. Email : benjamin.melinand@math.u-bordeaux.fr

[1]We consider that the centrifugal potential is constant and included in the pressure term.

$$\begin{cases} \mathcal{P}_{|z=\zeta} = P_0, \\ \partial_t \zeta - \underline{\mathbf{U}} \cdot \mathbf{N} = 0, \\ \mathbf{U}_b \cdot \mathbf{N}_b = 0, \end{cases} \tag{2}$$

where P_0 is constant, $\mathbf{N} = \begin{pmatrix} -\nabla\zeta \\ 1 \end{pmatrix}$, $\mathbf{N}_b = \begin{pmatrix} -\nabla b \\ 1 \end{pmatrix}$, $\underline{\mathbf{U}} = \begin{pmatrix} \underline{\mathbf{V}} \\ \underline{\mathrm{w}} \end{pmatrix} = \mathbf{U}_{|z=\zeta}$ and $\mathbf{U}_b = \begin{pmatrix} \mathbf{V}_b \\ \mathrm{w}_b \end{pmatrix} = \mathbf{U}_{|z=-H+b}$.

Influenced by the works of Zakharov ([37]) and Craig-Sulem-Sulem ([8]), Castro and Lannes in [5] shown that we can express the free surface Euler equations thanks to the unknowns $\left(\zeta, \mathbf{U}_{/\!/}, \boldsymbol{\omega}\right)$[2] where

$$\mathbf{U}_{/\!/} = \underline{\mathbf{V}} + \underline{\mathrm{w}} \nabla \zeta,$$

and $\boldsymbol{\omega}$ is the vorticity of the fluid. Then, they gave a system of three equations on these unknowns. In [26] we proceeded as Castro and Lannes and, taking into account the Coriolis force, we got the following system, called the Castro-Lannes system or the water waves equations,

$$\begin{cases} \partial_t \zeta - \underline{\mathbf{U}} \cdot \mathbf{N} = 0, \\ \partial_t \mathbf{U}_{/\!/} + \nabla\zeta + \frac{1}{2} \nabla \left| \mathbf{U}_{/\!/} \right|^2 - \frac{1}{2} \nabla \left[\left(1 + |\nabla\zeta|^2 \right) \underline{\mathrm{w}}^2 \right] + \underline{\boldsymbol{\omega}} \cdot \mathbf{N} \, \underline{\mathbf{V}}^{\perp} + f \underline{\mathbf{V}}^{\perp} = 0, \\ \partial_t \boldsymbol{\omega} + \left(\mathbf{U} \cdot \nabla_{X,z} \right) \boldsymbol{\omega} = \left(\boldsymbol{\omega} \cdot \nabla_{X,z} \right) \mathbf{U} + f \partial_z \mathbf{U}, \end{cases} \tag{3}$$

where $\underline{\boldsymbol{\omega}} = \boldsymbol{\omega}_{|z=\zeta}$ and $\mathbf{U} = \begin{pmatrix} \mathbf{V} \\ \mathrm{w} \end{pmatrix} = \mathbf{U}[\zeta, b](\mathbf{U}_{/\!/}, \boldsymbol{\omega})$ is the unique solution in $H^1(\Omega_t)$ of

$$\begin{cases} \mathrm{curl}\ \mathbf{U} = \boldsymbol{\omega} \text{ in } \Omega_t, \\ \mathrm{div}\ \mathbf{U} = 0 \text{ in } \Omega_t, \\ \left(\underline{\mathbf{V}} + \underline{\mathrm{w}} \nabla \zeta \right)_{|z=\zeta} = \mathbf{U}_{/\!/}, \\ \mathbf{U}_b \cdot \mathbf{N}_b = 0, \end{cases} \tag{4}$$

and with the following constraint

$$\nabla^{\perp} \cdot \mathbf{U}_{/\!/} = \underline{\boldsymbol{\omega}} \cdot \mathbf{N}. \tag{5}$$

Our principal motivation is the study of the long waves or Boussinesq regime. Hence, we nondimensionalize the previous equations. We have six physical parameters in our problem : the typical amplitude of the surface a, the typical amplitude of the bathymetry a_{bott}, the typical longitudinal scale L_x, the typical transverse scale L_y, the characteristic water depth H and the typical Coriolis frequency f. Then we can introduce five dimensionless parameters

[2] In fact, Castro and Lannes used the unknowns $\left(\zeta, \frac{\nabla}{\Delta} \cdot \mathbf{U}_{/\!/}, \boldsymbol{\omega}\right)$. But the unknowns $\left(\zeta, \mathbf{U}_{/\!/}, \boldsymbol{\omega}\right)$ are better to derive shallow water asymptotic models.

$$\varepsilon = \frac{a}{H},\ \beta = \frac{a_{\text{bott}}}{H},\ \mu = \frac{H^2}{L_x^2},\ \gamma = \frac{L_x}{L_y} \text{ and } \mathrm{Ro} = \frac{a\sqrt{gH}}{HfL_x}.$$

The parameter ε is called the nonlinearity parameter, β is called the bathymetric parameter, μ is called the shallowness parameter, γ is called the transversality parameter and Ro is the Rossby number. Then, we can nondimensionalize the Euler equations (1) and the Castro-Lannes equations (3) (see Part 1.2).

We organize our paper in four parts. In Subsection 1.2, we nondimensionalize the Castro-Lannes equations (see System (14)) and we give in Subsection 1.3 a local wellposedness result on these equations by taking into account the dependence on the dimensionless parameters. Section 2 is devoted to derive a generalization of the Boussinesq equations in 1D under a Coriolis forcing and to fully justify it. The Boussinesq equations are obtained under the assumption that μ is small, $\varepsilon, \beta = \mathcal{O}(\mu)$ (Boussinesq regime) and by neglecting all the terms of order $\mathcal{O}(\mu^2)$ in the nondimensionalized Euler equations or the water waves equations (see for instance [1] in the irrotational framework). It is a system of two equations on the free surface ζ and the vertical average of the horizontal component of the velocity denoted $\overline{\mathbf{V}} = (\overline{u}, \overline{v})^t$ (defined in (22)). Our Boussinesq-Coriolis equations are a system of three equations on the surface ζ, the average vertical velocity $\overline{\mathbf{V}}$ and the quantity $\mathbf{V}^\sharp = (u^\sharp, v^\sharp)^t$ (defined in (29)) which is introduced to catch interactions between the vorticity and the averaged velocity. These equations are the following system

$$\begin{cases} \partial_t \zeta + \partial_x \left([1 + \varepsilon\zeta - \beta b] \overline{u} \right) = 0, \\ \left(1 - \frac{\mu}{3}\partial_x^2\right) \partial_t \overline{u} + \partial_x \zeta + \varepsilon \overline{u} \partial_x \overline{u} - \frac{\varepsilon}{\mathrm{Ro}} \overline{v} + \frac{\varepsilon}{\mathrm{Ro}} \mu^{\frac{3}{2}} \frac{1}{24} \partial_x^2 \frac{v^\sharp}{h} = 0, \\ \partial_t \overline{v} + \varepsilon \overline{u} \partial_x \overline{v} + \frac{\varepsilon}{\mathrm{Ro}} \overline{u} = 0, \\ \partial_t \frac{\mathbf{V}^\sharp}{h} + \varepsilon \overline{u} \partial_x \frac{\mathbf{V}^\sharp}{h} + \frac{\varepsilon}{\mathrm{Ro}} \frac{\mathbf{V}^{\sharp\,\perp}}{h} = 0, \end{cases}$$

where $h = 1 + \varepsilon\zeta - \beta b$. Then, in Section 3 we derive and fully justify different asymptotic models in the Boussinesq regime when the bottom is flat. We first derive in Subsection 3.1 a linear system (System (38)) linked to the Klein-Gordon equation admitting the so-called Poincaré waves. Then, in Subsection 3.2 we study the Ostrovsky equation

$$\partial_\xi \left(\partial_\tau k + \frac{3}{2} k \partial_\xi k + \frac{1}{6} \partial_\xi^3 k \right) = \frac{1}{2} k.$$

This equation, derived by Ostrovsky ([27]), is a generalization of the KdV equation in presence of a Coriolis forcing. We offer a rigorous justification of the Ostrovsky approximation under a weak Coriolis forcing, i.e $\frac{\varepsilon}{\mathrm{Ro}} = \mathcal{O}(\sqrt{\mu})$. Notice that this work provides the first mathematical justification of the Ostrovsky equation. In Subsection 3.3 we fully justify the KdV approximation (equation (50)) when the rotation is very weak, i.e when $\frac{\varepsilon}{\mathrm{Ro}} = \mathcal{O}(\mu)$. Finally, in Section 4 we derive a generalization of the Green-Naghdi equations (62) in 1D under a Coriolis forcing with small bottom variations and we show that this system is consistent with the water waves equations. The Green-Naghdi equations are originally obtained in the irrotational framework under the assumption that μ is small and by neglecting all the terms of order $\mathcal{O}(\mu^2)$ in the nondimensionalized Euler equations or the water waves equations (see for instance [32] or Part 5.1.1.2 in [17] for a derivation in the irrotational framework). These equations were generalized in [4] in the rotational setting but without a Coriolis forcing. We add one in the paper.

1.1 Notations

- If $\mathbf{A} \in \mathbb{R}^3$, we denote by $\mathbf{A}_h$ its horizontal component.

- If $\mathbf{V} = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2$, we define the orthogonal of $\mathbf{V}$ by $\mathbf{V}^{\perp} = \begin{pmatrix} -v \\ u \end{pmatrix}$.

- In this paper, $C(\cdot)$ is a nondecreasing and positive function whose exact value has no importance.

- Consider a vector field $\mathbf{A}$ or a function w defined on Ω. Then, we denote $\underline{\mathbf{A}} = \mathbf{A}_{|z=\varepsilon\zeta}$, $\underline{\mathrm{w}} = \mathrm{w}_{|z=\varepsilon\zeta}$ and $\mathbf{A}_b = \mathbf{A}_{|z=-1+\beta b}$, $\mathrm{w}_b = \mathrm{w}_{|z=-1+\beta b}$.

- If $s \in \mathbb{R}$ and f is a function on $\mathbb{R}^2$, $|f|_{H^s}$ is its H^s-norm, $|f|_2$ is its L^2-norm and $|f|_{L^\infty}$ its $L^\infty(\mathbb{R}^2)$-norm.

- The operator $(\ ,\ )_2$ is the L^2-scalar product in $\mathbb{R}^2$.

- If f is a function defined on $\mathbb{R}^2$, we denote ∇f the gradient of f.

- If w is a function defined on Ω, $\nabla_{X,z}\mathrm{w}$ is the gradient of w and $\nabla_X \mathrm{w}$ its horizontal component.

- If $u = u(X, z)$ is defined in Ω, we define

$$\overline{u}(X) = \frac{1}{1 + \varepsilon\zeta - \beta b} \int_{-1+\beta b(X)}^{\varepsilon\zeta(X)} u(X, z) dz \text{ and } u^* = u - \overline{u}.$$

1.2 Nondimensionalization and the Castro-Lannes formulation

We recall the five dimensionless parameter

$$\varepsilon = \frac{a}{H},\ \beta = \frac{a_{\mathrm{bott}}}{H},\ \mu = \frac{H^2}{L_x^2},\ \gamma = \frac{L_x}{L_y} \text{ and } \mathrm{Ro} = \frac{a\sqrt{gH}}{HfL_x}. \tag{6}$$

We nondimensionalize the variables and the unknowns. We introduce (see [17] or [26] for instance for an explanation of this nondimensionalization)

$$\begin{cases} x' = \dfrac{x}{L_x},\ y' = \dfrac{y}{L_y},\ z' = \dfrac{z}{H},\ \zeta' = \dfrac{\zeta}{a},\ b' = \dfrac{b}{a_{\mathrm{bott}}},\ t' = \dfrac{\sqrt{gH}}{L_x} t, \\ \mathbf{V}' = \sqrt{\dfrac{H}{g}} \dfrac{\mathbf{V}}{a},\ \mathrm{w}' = H\sqrt{\dfrac{H}{g}} \dfrac{\mathrm{w}}{aL_x} \text{ and } \mathcal{P}' = \dfrac{\mathcal{P}}{\rho g H}. \end{cases} \tag{7}$$

In this paper, we use the following notations

$$\nabla^\gamma = \nabla^\gamma_{X'} = \begin{pmatrix} \partial_{x'} \\ \gamma\partial_{y'} \end{pmatrix}, \quad = \nabla^{\mu,\gamma}_{X',z'} = \begin{pmatrix} \sqrt{\mu}\nabla^\gamma_{X'} \\ \partial_{z'} \end{pmatrix}, \mathrm{curl}^{\mu,\gamma} = \nabla^{\mu,\gamma}_{X',z'} \times , \mathrm{div}^{\mu,\gamma} = \nabla^{\mu,\gamma}_{X',z'} \cdot . \quad (8)$$

We also define

$$\mathbf{U}^\mu = \begin{pmatrix} \sqrt{\mu}\mathbf{V}' \\ \mathrm{w}' \end{pmatrix}, \boldsymbol{\omega}' = \frac{1}{\mu}\mathrm{curl}^{\mu,\gamma}\mathbf{U}^\mu, \underline{\mathbf{U}}^\mu = \begin{pmatrix} \sqrt{\mu}\underline{\mathbf{V}}' \\ \underline{\mathrm{w}}' \end{pmatrix} = \mathbf{U}^\mu_{|z'=\varepsilon\zeta'}, \mathbf{U}^\mu_b = \mathbf{U}^\mu_{|z'=-1+\beta b'}, \quad (9)$$

and

$$\mathbf{N}^{\mu,\gamma} = \begin{pmatrix} -\varepsilon\sqrt{\mu}\nabla^\gamma\zeta' \\ 1 \end{pmatrix}, \mathbf{N}^{\mu,\gamma}_b = \begin{pmatrix} -\beta\sqrt{\mu}\nabla^\gamma b' \\ 1 \end{pmatrix}. \quad (10)$$

Notice that our nondimensionalization of the vorticity allows us to consider only weakly sheared flows (see [4], [34], [30]). The nondimensionalized fluid domain is

$$\Omega'_{t'} := \{(X', z') \in \mathbb{R}^3 \ , \ -1 + \beta b'(X') < z' < \varepsilon\zeta'(t', X')\}. \quad (11)$$

Finally, if $\mathbf{V} = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2$, we define $\mathbf{V}$ by $\mathbf{V}^\perp = \begin{pmatrix} -v \\ u \end{pmatrix}$. Then, the Euler-Coriolis equations (1) become

$$\begin{cases} \partial_{t'}\mathbf{U}^\mu + \frac{\varepsilon}{\mu}\left(\mathbf{U}^\mu \cdot \nabla^{\mu,\gamma}_{X',z'}\right)\mathbf{U}^\mu + \frac{\varepsilon\sqrt{\mu}}{\mathrm{Ro}}\begin{pmatrix} \mathbf{V}'^\perp \\ 0 \end{pmatrix} = -\frac{1}{\varepsilon}\nabla^{\mu,\gamma}_{X',z'}\mathcal{P}' - \frac{1}{\varepsilon}\boldsymbol{e_z} \text{ in } \Omega'_t, \\ \mathrm{div}^{\mu,\gamma}_{X',z'}\ \mathbf{U}^\mu = 0 \text{ in } \Omega'_t, \end{cases} \quad (12)$$

with the boundary conditions

$$\begin{cases} \partial_{t'}\zeta' - \frac{1}{\mu}\underline{\mathbf{U}}^\mu \cdot \mathbf{N}^{\mu,\gamma} = 0, \\ \mathbf{U}^\mu_b \cdot \mathbf{N}^{\gamma,\mu}_b = 0. \end{cases} \quad (13)$$

We can also nondimensionalize the Castro-Lannes formulation. We introduce the quantity

$$\mathbf{U}^{\mu,\gamma}_{/\!/} = \underline{\mathbf{V}} + \varepsilon\underline{\mathrm{w}}\nabla^\gamma\zeta.$$

Then, the Castro-Lannes formulation becomes (see [5] or [26] when $\gamma = 1$),

$$\begin{cases} \partial_t\zeta - \frac{1}{\mu}\underline{\mathbf{U}}^\mu \cdot \mathbf{N}^{\mu,\gamma} = 0, \\ \partial_t\mathbf{U}^{\mu,\gamma}_{/\!/} + \nabla^\gamma\zeta + \frac{\varepsilon}{2}\nabla^\gamma\left|\mathbf{U}^{\mu,\gamma}_{/\!/}\right|^2 - \frac{\varepsilon}{2\mu}\nabla^\gamma\left[\left(1 + \varepsilon^2\mu\,|\nabla^\gamma\zeta|^2\right)\underline{\mathrm{w}}^2\right] + \varepsilon\underline{\boldsymbol{\omega}} \cdot \mathbf{N}^{\mu,\gamma}\,\underline{\mathbf{V}}^\perp + \frac{\varepsilon}{\mathrm{Ro}}\underline{\mathbf{V}}^\perp = 0, \\ \partial_t\boldsymbol{\omega} + \frac{\varepsilon}{\mu}\left(\mathbf{U}^\mu \cdot \nabla^{\mu,\gamma}_{X,z}\right)\boldsymbol{\omega} = \frac{\varepsilon}{\mu}\left(\boldsymbol{\omega} \cdot \nabla^{\mu,\gamma}_{X,z}\right)\mathbf{U}^\mu + \frac{\varepsilon}{\mu\mathrm{Ro}}\partial_z\mathbf{U}^\mu, \end{cases} \quad (14)$$

where $\mathbf{U}^\mu = \begin{pmatrix} \sqrt{\mu}\mathbf{V} \\ \mathrm{w} \end{pmatrix} = \mathbf{U}^\mu[\varepsilon\zeta, \beta b](\mathbf{U}_{/\!/}^{\mu,\gamma}, \boldsymbol{\omega})$ is the unique solution in $H^1(\Omega_t)$ of

$$\begin{cases} \mathrm{curl}^{\mu,\gamma}\, \mathbf{U}^\mu = \mu\boldsymbol{\omega} \text{ in } \Omega_t, \\ \mathrm{div}^{\mu,\gamma}\, \mathbf{U}^\mu = 0 \text{ in } \Omega_t, \\ (\underline{\mathbf{V}} + \varepsilon\underline{\mathrm{w}}\nabla^\gamma\zeta)_{|z=\varepsilon\zeta} = \mathbf{U}_{/\!/}^{\mu,\gamma}, \\ \mathbf{U}_b^\mu \cdot \mathbf{N}_b^{\mu,\gamma} = 0, \end{cases} \tag{15}$$

and with the following constraint

$$\nabla^\perp \cdot \mathbf{U}_{/\!/}^{\mu,\gamma} = \underline{\boldsymbol{\omega}} \cdot \mathbf{N}^{\mu,\gamma}. \tag{16}$$

Remark 1.1. *When, $\boldsymbol{\omega} = 0$ and $Ro = +\infty$, we get the irrotational water waves equations (see Remark 2.4 in [5]). In particular in this situation, when $\gamma = 0$ we can check that the velocity $\mathbf{U}^\mu$ becomes two dimensional : $\mathbf{U}^\mu = \left(\sqrt{\mu}\mathbf{V}_x, 0, \mathrm{w}\right)^t$. This is not the case when $\boldsymbol{\omega} \neq 0$. Even if $\gamma = 0$, the vorticity transfers energy from $\mathbf{V}_x$ to $\mathbf{V}_y$. The only way to get a two dimensional speed is to assume that $\omega = (0, \omega_y, 0)^t$ (see for instance [18]).*

Remark 1.2. *Notice that if $\left(\zeta, \mathbf{U}_{/\!/}^{\mu,\gamma}, \boldsymbol{\omega}\right)$ is a solution of the Castro-Lannes system* (14)*, $\nabla^\perp \cdot \mathbf{U}_{/\!/}^{\mu,\gamma}$ satisfies the equation*

$$\partial_t \nabla^\perp \cdot \mathbf{U}_{/\!/}^{\mu,\gamma} + \nabla^\gamma \cdot \left(\varepsilon\underline{\boldsymbol{\omega}} \cdot \mathbf{N}^{\mu,\gamma}\underline{\mathbf{V}}^\perp + \frac{\varepsilon}{\mathrm{Ro}}\underline{\mathbf{V}}\right) = 0.$$

Furthermore, by taking the trace of the third equation of the Castro-Lannes system (14)*, we can see that $\underline{\boldsymbol{\omega}} \cdot \mathbf{N}^{\mu,\gamma}$ satisfies the equation*

$$\partial_t \left(\underline{\boldsymbol{\omega}} \cdot \mathbf{N}^{\mu,\gamma}\right) + \nabla^\gamma \cdot \left(\varepsilon\underline{\boldsymbol{\omega}} \cdot \mathbf{N}^{\mu,\gamma}\underline{\mathbf{V}}^\perp + \frac{\varepsilon}{\mathrm{Ro}}\underline{\mathbf{V}}\right) = 0,$$

Hence, the constraint (16) *is propagated by the equations.*

We add a technical assumption. We assume that the water depth is bounded from below by a positive constant

$$\exists\, h_{\min} > 0 \;,\; 1 + \varepsilon\zeta - \beta b \geq h_{\min}. \tag{17}$$

We also suppose that the dimensionless parameters satisfy

$$\exists \mu_{\max},\; 0 < \mu \leq \mu_{\max},\; 0 < \varepsilon \leq 1,\; 0 \leq \gamma \leq 1,\; 0 \leq \beta \leq 1 \text{ and } \frac{\varepsilon}{\mathrm{Ro}} \leq 1. \tag{18}$$

Remark 1.3. *We have $\frac{\varepsilon}{Ro} = \frac{fL}{\sqrt{gH}}$. As said in [26], it is quite reasonable to assume that $\frac{\varepsilon}{Ro} \leq 1$ since for water waves, the typical rotation speed due to the Coriolis forcing is less than the typical water wave celerity (see for instance [29], [11], [20]).*

1.3 Useful results

In this paper, we fully justify different asymptotic models of the water waves equations. Then, we have to define the notion of consistence (see for instance [17]).

Definition 1.4. *The Castro-Lannes equations* (14) *are consistent of order* $\mathcal{O}\left(\mu^k\right)$ *with a system of equations* S *for* ζ *and* $\overline{\mathbf{V}}$ *if for all sufficiently smooth solutions* $\left(\zeta, \mathbf{U}_{/\!/}^{\mu,\gamma}, \boldsymbol{\omega}\right)$ *of the Castro-Lannes equations* (14) *, the pair* $\left(\zeta, \overline{\mathbf{V}}[\varepsilon\zeta, \beta b]\left(\mathbf{U}_{/\!/}^{\mu,\gamma}, \boldsymbol{\omega}\right)\right)$ (defined in (22)) *solves* S *up to a residual of order* $\mathcal{O}\left(\mu^k\right)$*.*

We also need an existence result for the Castro-Lannes formulation (14). This is the purpose of the next theorem proven in [26]. We recall that the existence of the water waves equations is always under the so-called Rayleigh-Taylor condition assuming the positivity of the Rayleigh-Taylor coefficient $\mathfrak{a}$ (see Part 3.4.5 in [17] for the link between $\mathfrak{a}$ and the Rayleigh-Taylor condition or [26]) where

$$\mathfrak{a} := \mathfrak{a}[\varepsilon\zeta, \beta b](\mathbf{U}_{/\!/}^{\mu,\gamma}, \boldsymbol{\omega}) = 1 + \varepsilon\left(\partial_t + \varepsilon\underline{\mathbf{V}}[\varepsilon\zeta, \beta b](\mathbf{U}_{/\!/}^{\mu,\gamma}, \boldsymbol{\omega}) \cdot \nabla\right)\underline{\mathrm{w}}[\varepsilon\zeta, \beta b](\mathbf{U}_{/\!/}^{\mu,\gamma}, \boldsymbol{\omega}). \quad (19)$$

Notice that in [26] we explain how we can define initially the Rayleigh-Taylor coefficient $\mathfrak{a}$.

Theorem 1.5. *Let* $A > 0$, $\mathbf{N} \geq 5$, $b \in H^{N+2}(\mathbb{R}^2)$. We assume that

$$\left(\zeta_0, (\mathbf{U}_{/\!/}^{\mu,\gamma})_0, \boldsymbol{\omega}_0\right) \in H^{N+\frac{1}{2}}(\mathbb{R}^2) \times H^N(\mathbb{R}^2) \times H^{N-1}(\Omega_0),$$

that $\nabla^{\mu,\gamma} \cdot \boldsymbol{\omega}_0 = 0$ *and that Condition* (16) *is satisfied. We suppose that* $(\varepsilon, \beta, \gamma, \mu, \mathrm{Ro})$ *satisfy* (18)*. Finally, we assume that*

$$\exists\, h_{\min}, \mathfrak{a}_{\min} > 0\ ,\ \varepsilon\zeta_0 + 1 - \beta b \geq h_{\min} \text{ and } \mathfrak{a}[\varepsilon\zeta_0, \beta b]((\mathbf{U}_{/\!/}^{\mu,\gamma})_0, \boldsymbol{\omega}_0) \geq \mathfrak{a}_{\min},$$

and

$$|\zeta_0|_{H^{N+\frac{1}{2}}} + \left|\frac{1}{\sqrt{1+\sqrt{\mu}|D|}}(\mathbf{U}_{/\!/}^{\mu,\gamma})_0\right|_{H^N} + ||\boldsymbol{\omega}_0||_{H^{N-1}} \leq A.$$

Then, there exists $T > 0$ *and a unique classical solution* $\left(\zeta, \mathbf{U}_{/\!/}^{\mu,\gamma}, \boldsymbol{\omega}\right)$ *to the Castro-Lannes* (14) *with initial data* $\left(\zeta_0, (\mathbf{U}_{/\!/}^{\mu,\gamma})_0, \boldsymbol{\omega}_0\right)$*. Moreover,*

$$T = \frac{T_0}{\max(\varepsilon, \beta, \frac{\varepsilon}{\mathrm{Ro}})}\ ,\ \frac{1}{T_0} = c^1,$$
$$\max_{[0,T]}\left(|\zeta(t,\cdot)|_{H^N} + \left|\frac{1}{\sqrt{1+\sqrt{\mu}|D|}}\mathbf{U}_{/\!/}^{\mu,\gamma}(t,\cdot)\right|_{H^{N-\frac{1}{2}}} + ||\boldsymbol{\omega}(t,\cdot)||_{H^{N-1}}\right) = c^2,$$

with $c^j = C\left(A, \mu_{\max}, \frac{1}{h_{\min}}, \frac{1}{\mathfrak{a}_{\min}}, |b|_{H^{N+2}}\right)$.

Thanks to this theorem, we know that the quantities ζ, $\mathbf{U}_{/\!/}^{\mu,\gamma}$, $\boldsymbol{\omega}$ and then $\overline{\mathbf{V}}$ (defined in (22)) remain bounded uniformly with respect to the small parameters during the time evolution of the flow, which will be essential to derive rigorously asymptotic models.

2 Boussinesq-Coriolis equations when $\gamma = 0$

This part is devoted to the derivation and the full justification of the Boussinesq-Coriolis equations (31) under a Coriolis forcing and with $\gamma = 0$. These equations are an order $\mathcal{O}(\mu^2)$ approximation of the water waves equations under the assumption that $\varepsilon, \beta = \mathcal{O}(\mu)$. The corresponding regime is called *long wave regime* or Boussinesq regime. Contrary to [4], whose approach is based on the averaged Euler equations, our derivation is based on the Castro-Lannes equations (14). Then, the asymptotic regime is

$$\mathcal{A}_{\text{Bouss}} = \left\{ (\varepsilon, \beta, \gamma, \mu, \text{Ro}), 0 \leq \mu \leq \mu_0, \frac{\varepsilon}{\text{Ro}} \leq 1, \varepsilon = \mathcal{O}(\mu), \beta = \mathcal{O}(\mu), \gamma = 0 \right\}. \tag{20}$$

Remark 2.1. *In fact, we can relax the assumption $\gamma = 0$ by only assuming that $\gamma = \mathcal{O}(\mu^2)$ since we neglect all the terms of order $\mathcal{O}(\mu^2)$ in the following.*

We introduce the water depth

$$h(t, X) = 1 + \varepsilon\zeta(t, X) - \beta b(X), \tag{21}$$

and the averaged horizontal velocity

$$\overline{\mathbf{V}} = \overline{\mathbf{V}}[\varepsilon\zeta, \beta b](\mathbf{U}_{/\!/}^{\mu,\gamma}, \boldsymbol{\omega})(t, X) = \frac{1}{h(t, X)} \int_{z=-1+\beta b(X)}^{\varepsilon\zeta(t,X)} \mathbf{V}[\varepsilon\zeta, \beta b](\mathbf{U}_{/\!/}^{\mu,\gamma}, \boldsymbol{\omega})(t, X, z) dz. \tag{22}$$

More generally, if u is a function defined in Ω, $\overline{u}$ is its average and $u^* = u - \overline{u}$. In the following we denote $\mathbf{V} = (u, v)^t$. As noticed in [5], we have to introduce the "shear" velocity

$$\mathbf{V}_{\text{sh}} = \mathbf{V}_{\text{sh}}[\varepsilon\zeta, \beta b](\mathbf{U}_{/\!/}^{\mu,\gamma}, \boldsymbol{\omega})(t, X) = (u_{\text{sh}}, v_{\text{sh}}) = \int_z^{\varepsilon\zeta} \boldsymbol{\omega}_h^{\perp} \tag{23}$$

and its average

$$\mathbf{Q} = \left(\text{Q}_x, \text{Q}_y\right)^t = \overline{\mathbf{V}}_{\text{sh}} = \frac{1}{h} \int_{-1+\beta b}^{\varepsilon\zeta} \int_{z'}^{\varepsilon\zeta} \boldsymbol{\omega}_h^{\perp}.$$

When $\gamma = 0$, $\mathbf{U}_{/\!/}^{\mu,\gamma} = (\underline{u} + \varepsilon \underline{\mathrm{w}} \partial_x \zeta, \underline{v})^t$. Hence in the following, we denote

$$u_{/\!/} = \underline{u} + \varepsilon \underline{\mathrm{w}} \partial_x \zeta. \tag{24}$$

In this section, we do the asymptotic expansion with respect to μ of different quantities. In the following, we denote by R a remainder whose exact value has no importance and which is bounded uniformly with respect to μ.

Remark 2.2. *Notice that thanks to Theorem 1.5, we know that the quantities* ζ, $\mathbf{U}_{/\!/}^{\mu,\gamma}$, $\boldsymbol{\omega}$, $\overline{\mathbf{V}}$ and $\mathbf{U}$ *remain bounded uniformly with respect to the small parameters during the time evolution of the flow. Furthermore,* $\partial_t\zeta$, $\partial_t\mathbf{U}_{/\!/}^{\mu,\gamma}$, $\partial_t\boldsymbol{\omega}$ and $\partial_t\mathbf{U}$ *also remain bounded uniformly with respect to the small parameters during this time.*

2.1 Asymptotic expansion for the velocity and useful identities

In this part, we give an expansion of the velocity with respect to μ. First we recall the following fact (the proof is a small adaptation of Proposition 4.2 in [26]).

Proposition 2.3. *If* $\left(\zeta, \mathbf{U}_{/\!/}^{\mu,\gamma}, \boldsymbol{\omega}\right)$ *satisfy the Castro-Lannes system* (14)*, we have*

$$\underline{\mathbf{U}}^{\mu} \cdot \mathbf{N}^{\mu,\gamma} = -\mu\nabla^{\gamma} \cdot \left(h\overline{\mathbf{V}}\right).$$

This proposition, coupled with the first equation of (14), gives us an equation that links ζ to $\overline{\mathbf{V}}$. In particular, when $\gamma = 0$, we get an equation that links ζ to $\overline{u}$. We also need an expansion of u and v with respect to μ. The following proposition is for v.

Proposition 2.4. *If* $\left(\zeta, \mathbf{U}_{/\!/}^{\mu,0}, \boldsymbol{\omega}\right)$ *satisfy the Castro-Lannes system* (14)*, we have*

$$v = \overline{v} + \sqrt{\mu}v_{\mathrm{sh}}^{*},$$
$$\underline{v} = \overline{v} - \sqrt{\mu}\mathrm{Q}_y,$$
$$\underline{\boldsymbol{\omega}} \cdot \mathbf{N}^{\mu,0} = \partial_x\underline{v}$$

and

$$\partial_t\underline{v} + \varepsilon\underline{u}\partial_x\underline{v} + \frac{\varepsilon}{\mathrm{Ro}}\underline{u} = 0.$$

Proof. Since $\mathrm{curl}^{\mu,0}\,\mathbf{U}^{\mu} = \mu\boldsymbol{\omega}$, we get that

$$\sqrt{\mu}\boldsymbol{\omega}_x = -\partial_z v \text{ and } \boldsymbol{\omega}_z = \partial_x v. \tag{25}$$

Then, plugging the ansatz $v = \overline{v} + \sqrt{\mu}v_1$ in the first equation and using the fact that the average of v_1 is equal to 0 we get

$$\underline{v} = \overline{v} - \sqrt{\mu}\frac{1}{h}\int_{-1+\beta b}^{\varepsilon\zeta}\int_{z'}^{\varepsilon\zeta}\boldsymbol{\omega}_x.$$

Furthermore, from the equation on the second component of $\mathbf{U}_{/\!/}^{\mu,0}$, we have

$$\partial_t\underline{v} + \varepsilon\boldsymbol{\omega} \cdot \mathbf{N}^{\mu,0}\underline{u} + \frac{\varepsilon}{\mathrm{Ro}}\underline{u} = 0.$$

Then, using the second equation of (25), we get that $\underline{\boldsymbol{\omega}} \cdot \mathbf{N}^{\mu,0} = \partial_x\underline{v}$ and the result follows.

□

The expansion of u is more complex and also involves an expansion of w. It is the purpose of the following proposition. But before, we also have to introduce the following operators

$$T[\varepsilon\zeta,\beta b]f=\int_z^{\varepsilon\zeta}\partial_x^2\int_{-1+\beta b}^{z'}f \text{ and } T^*[\varepsilon\zeta,\beta b]f=(T[\varepsilon\zeta,\beta b]f)^*,$$

When no confusion is possible, we denote $T=T[\varepsilon\zeta,\beta b]$ and $T^*=T^*[\varepsilon\zeta,\beta b]$.

Proposition 2.5. *If* $\left(\zeta,\mathbf{U}_{/\!/}^{\mu,0},\boldsymbol{\omega}\right)$ *satisfy the Castro-Lannes system* (14), *we have*

$$u=\overline{u}+\sqrt{\mu}u_{\mathrm{sh}}^*+\mu T^*\overline{u}+\mu^{\frac{3}{2}}T^*u_{\mathrm{sh}}^*+\mu^2R,$$
$$\underline{u}=\overline{u}-\sqrt{\mu}\mathrm{Q}_x+\mu\underline{T^*\overline{u}}-\mu^{\frac{3}{2}}\overline{Tu_{\mathrm{sh}}^*}+\mu^2R,$$

where $T^*\overline{u}=-\frac{1}{2}\left([z+1-\beta b]^2-\frac{h^2}{3}\right)\partial_x^2\overline{u}+\beta R$. *We also have*

$$\mathrm{w}=-\mu\partial_x\left(\int_{-1+\beta b}^{z}u\right),$$
$$\underline{\mathrm{w}}=-\mu h\partial_x\overline{u}-\mu^{\frac{3}{2}}\partial_x h\mathrm{Q}_x+\max(\mu^2,\beta\mu)R,$$

and

$$u_{/\!/}=\overline{u}-\sqrt{\mu}\mathrm{Q}_x-\mu\frac{1}{3h}\partial_x\left(h^3\partial_x\overline{u}\right)-\mu^{\frac{3}{2}}\left(\overline{Tu_{\mathrm{sh}}^*}+\mathrm{Q}_x\left(\partial_x h\right)^2\right)+\max(\mu^2,\beta\mu)R.$$

Proof. This proof is a small adaptation of part 2.2 in [4] and Part 4.2 in [26]. We recall the main steps. Using the fact that the velocity is divergence free and Proposition 2.3, we get

$$\mathrm{w}=-\mu\partial_x\left(\int_{-1+\beta b}^{z}u\right).$$

Furthermore, since $\mathrm{curl}^{\mu,0}\,\mathbf{U}^\mu=\mu\boldsymbol{\omega}$, we get that

$$\sqrt{\mu}\boldsymbol{\omega}_y=\partial_z u-\partial_x\mathrm{w}.$$

Then, plugging the ansatz $u=\overline{u}+\sqrt{\mu}u_1$ and using the fact that the average of u_1 is zero, we get

$$u_1=-\left(\int_z^{\varepsilon\zeta}\boldsymbol{\omega}_y\right)^*-\frac{1}{\sqrt{\mu}}\left(\int_z^{\varepsilon\zeta}\partial_x\mathrm{w}\right)^*$$

and

$$u=\overline{u}+\sqrt{\mu}u_{\mathrm{sh}}^*+\mu T^*u. \tag{26}$$

Then, the expansion for u follows by applying $1+\mu T^*$ to the previous equation. Notice that $\underline{T^* u} = -\overline{Tu}$. The computation of $T^*\overline{u}$ follows from the fact that $\overline{u}$ does not depend on z. Finally, the expansion $\underline{\mathrm{w}}$ and $u_{/\!/}$ is the direct consequence for Proposition 2.3 and the expansion of u.

□

Thanks to the previous proposition, we can also get an expansion of $\partial_t u$ and $\partial_t \mathrm{w}$.

Proposition 2.6. *If* $\left(\zeta, \mathbf{U}_{/\!/}^{\mu,0}, \boldsymbol{\omega}\right)$ *satisfy the Castro-Lannes system* (14)*, we have*

$$\begin{aligned}
&\partial_t \left(u - \overline{u} - \sqrt{\mu} u_{\mathrm{sh}}^* - \mu T^* \overline{u} - \mu^{\frac{3}{2}} T^* u_{\mathrm{sh}}^*\right) = \mu^2 R, \\
&\partial_t \left(\underline{u} - \overline{u} + \sqrt{\mu} \mathrm{Q}_x - \mu \underline{T^* \overline{u}} + \mu^{\frac{3}{2}} \overline{T u_{\mathrm{sh}}^*}\right) = \mu^2 R, \\
&\partial_t \left(\underline{\mathrm{w}} + \mu h \partial_x \overline{u} + \mu^{\frac{3}{2}} \partial_x h \mathrm{Q}_x\right) = \max(\mu^2, \beta\mu) R.
\end{aligned} \tag{27}$$

Proof. From Equality (26) we get that

$$u = (1 - \mu T^*)\left(\overline{u} + \sqrt{\mu} u_{\mathrm{sh}}^*\right) + \mu^2 T^* T^* u. \tag{28}$$

Hence the first and the second equations follows from Remark 2.2. For the third equation, we get the result thanks to Proposition (2.3) and Remark 2.2. □

As [4] noticed, we can not express $\overline{T u_{\mathrm{sh}}^*}$ in terms of ζ and $\overline{\mathbf{V}}$. Then, we have to introduce

$$\begin{aligned}
\mathbf{V}^\sharp = (u^\sharp, v^\sharp)^t &= -\frac{24}{h^3} \int_{-1+\beta b}^{\varepsilon\zeta} \int_z^{\varepsilon\zeta} \int_{-1+\beta b}^{z} \left(u_{\mathrm{sh}}^*, v_{\mathrm{sh}}^*\right)^t, \\
&= \frac{12}{h^3} \int_{-1+\beta b}^{\varepsilon\zeta} (1 + z - \beta b)^2 \left(u_{\mathrm{sh}}^*, v_{\mathrm{sh}}^*\right)^t.
\end{aligned} \tag{29}$$

Notice that the previous equality follows from a double integration by parts. We have the following Lemma.

Lemma 2.7. *We have the following equalities*

$$\begin{aligned}
\overline{T u_{sh}^*} &= -\left(\varepsilon \partial_x \zeta\right)^2 Q_x + \frac{1}{h} \int_{-1+\beta b}^{\varepsilon\zeta} \partial_x \int_z^{\varepsilon\zeta} \partial_x \int_{-1+\beta b}^{z} u_{sh}^* \\
&= -\left(\partial_x h\right)^2 Q_x - \frac{1}{24h} \partial_x^2 \left(h^3 u^\sharp\right) + \beta R.
\end{aligned}$$

Proof. We have

$$\partial_x \int_z^{\varepsilon\zeta} \partial_x \int_{-1+\beta b}^{z} u_{\mathrm{sh}}^* = \int_z^{\varepsilon\zeta} \partial_x^2 \int_{-1+\beta b}^{z} u_{\mathrm{sh}}^* + \underline{\varepsilon \partial_x \zeta \partial_x \int_{-1+\beta b}^{z} u_{\mathrm{sh}}^*} \tag{30}$$

and the first equality follows from the fact that the average of u_{sh}^* is zero and that $\underline{u_{\mathrm{sh}}^*} = -\mathrm{Q}_x$. The second equality follows from the same arguments. □

In the following section, we give equations for Q_x, Q_y $\mathbf{V}^\sharp$ since we can not express these quantities with respect to ζ and $\overline{\mathbf{V}}$. These equations are essential to derive the Boussinesq-Coriolis equations.

2.2 Equations for Q_x, Q_y and $\mathbf{V}^\sharp$

In this part we give the equations satisfied by Q_x and Q_y at order $\mathcal{O}\left(\mu^{\frac{3}{2}}\right)$. The computations are similar to Part 5.4.1 in [4]. We start by Q_x.

Proposition 2.8. *If* $\left(\zeta, \mathbf{U}_{/\!/}^{\mu,0}, \boldsymbol{\omega}\right)$ *satisfy the Castro-Lannes system* (14)*, then, in the Boussinesq regime* (20)*, Q_x satisfies the following equation*

$$\partial_t Q_x + \varepsilon \overline{u} \partial_x Q_x + \varepsilon Q_x \partial_x \overline{u} + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}} \left(\underline{v} - \overline{v}\right) = \mu^{\frac{3}{2}} R,$$

and u^*_{sh} satisfies the equation

$$\partial_t u^*_{\mathrm{sh}} + \varepsilon \overline{u} \partial_x u^*_{\mathrm{sh}} + \varepsilon u^*_{\mathrm{sh}} \partial_x \overline{u} + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}} \left(\overline{v} - v\right) = \mu^{\frac{3}{2}} R.$$

Proof. Using the second equation of the vorticity equation of the Castro-Lannes system (14), we have

$$\partial_t \boldsymbol{\omega}_y + \varepsilon u \partial_x \boldsymbol{\omega}_y + \frac{\varepsilon}{\mu} \mathrm{w} \partial_z \boldsymbol{\omega}_y = \varepsilon \boldsymbol{\omega}_x \partial_x v + \frac{\varepsilon}{\sqrt{\mu}} \boldsymbol{\omega}_z \partial_z v + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}} \partial_z v.$$

Since $\boldsymbol{\omega}_x = -\frac{1}{\sqrt{\mu}} \partial_z v$ and $\boldsymbol{\omega}_z = \partial_x v$ we notice that $\varepsilon \boldsymbol{\omega}_x \partial_x v + \frac{\varepsilon}{\sqrt{\mu}} \boldsymbol{\omega}_z \partial_z v = 0$. Using Proposition 2.5 we get

$$\partial_t \boldsymbol{\omega}_y + \varepsilon \overline{u} \partial_x \boldsymbol{\omega}_y - \varepsilon \partial_x \left[(1 + z - \beta b) \overline{u}\right] \partial_z \boldsymbol{\omega}_y - \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}} \partial_z v = \mu^{\frac{3}{2}} R,$$

Then, integrating with respect to z, using the fact that $\partial_t \zeta + \partial_x \left(h \overline{u}\right) = 0$ and $u_{\mathrm{sh}} = -\int_z^{\varepsilon\zeta} \boldsymbol{\omega}_y$, we get

$$\partial_t u_{\mathrm{sh}} + \varepsilon \overline{u} \partial_x u_{\mathrm{sh}} + \varepsilon u_{\mathrm{sh}} \partial_x \overline{u} + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}} \left(\underline{v} - v\right) = \varepsilon \partial_x \left[\left(1 + z - \beta b\right) \overline{u}\right] \partial_z u_{\mathrm{sh}} + \mu^{\frac{3}{2}} R.$$

Integrating again with respect to z, using the fact that $\partial_t \zeta + \partial_x \left(h \overline{u}\right) = 0$ and $Q_x = \overline{u^*_{\mathrm{sh}}}$, we obtain

$$\partial_t Q_x + \varepsilon \overline{u} \partial_x Q_x + \varepsilon Q_x \partial_x \overline{u} + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}} \left(\underline{v} - \overline{v}\right) = \mu^{\frac{3}{2}} R.$$

□

We have a similar equation for Q_y.

Proposition 2.9. *If* $\left(\zeta, \mathbf{U}_{/\!/}^{\mu,0}, \boldsymbol{\omega}\right)$ *satisfy the Castro-Lannes system* (14)*, then, in the Boussinesq regime* (20)*,* Q_x *satisfies the following equation*

$$\partial_t \mathrm{Q}_y + \varepsilon \overline{u} \partial_x \mathrm{Q}_y + \varepsilon \mathrm{Q}_x \partial_x \overline{v} + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}} \left(\overline{u} - \underline{u}\right) = \mu^{\frac{3}{2}} R$$

and v_{sh}^* *satisfies the equation*

$$\partial_t v_{\mathrm{sh}}^* + \varepsilon \overline{u} \partial_x v_{\mathrm{sh}}^* + \varepsilon u_{\mathrm{sh}}^* \partial_x \overline{v} + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}} \left(u - \overline{u}\right) = \mu^{\frac{3}{2}} R.$$

Proof. Using the first equation of the vorticity equation of the Castro-Lannes system (14), we have

$$\partial_t \boldsymbol{\omega}_x + \varepsilon u \partial_x \boldsymbol{\omega}_x + \frac{\varepsilon}{\mu} \mathrm{w} \partial_z \boldsymbol{\omega}_x = \varepsilon \boldsymbol{\omega}_x \partial_x u + \frac{\varepsilon}{\sqrt{\mu}} \boldsymbol{\omega}_z \partial_z u + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}} \partial_z u.$$

Then, using the fact that $\nabla^{\mu,0} \cdot \boldsymbol{\omega} = 0$ and $\nabla^{\mu,0} \cdot \mathbf{U}^{\mu,\gamma} = 0$, we get

$$\partial_t \boldsymbol{\omega}_x - \frac{\varepsilon}{\sqrt{\mu}} \partial_z \left(u \boldsymbol{\omega}_z\right) + \frac{\varepsilon}{\mu} \partial_z \left(\mathrm{w} \boldsymbol{\omega}_x\right) = \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}} \partial_z u.$$

then, we integrate with respect to z and, using the fact that $\partial_t \zeta - \frac{1}{\mu} \underline{\mathbf{U}}^{\mu} \cdot \mathbf{N}^{\mu,0} = 0$, $\boldsymbol{\omega}_x = -\frac{1}{\sqrt{\mu}} \partial_z v$ and $\boldsymbol{\omega}_z = \partial_x v$, we obtain

$$\partial_t \left(\int_{-1+\beta b}^{\varepsilon \zeta} \boldsymbol{\omega}_x \right) - \frac{\varepsilon}{\sqrt{\mu}} \underline{u} \partial_x \underline{v} + \frac{\varepsilon}{\sqrt{\mu}} u \partial_x v + \frac{\varepsilon}{\mu^{\frac{3}{2}}} \mathrm{w} \partial_z v + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}} \left(u - \underline{u}\right) = 0.$$

Then, we integrate again with respect to z and, using Proposition 2.4 and the fact that $\partial_t \zeta - \frac{1}{\mu} \underline{\mathbf{U}}^{\mu} \cdot \mathbf{N}^{\mu,0} = 0$, $\mathbf{U}_b^{\mu} \cdot \mathbf{N}_b^{\mu,0} = 0$, and $\nabla^{\mu,0} \cdot \mathbf{U}^{\mu} = 0$, we get

$$\partial_t \mathrm{Q}_y - \frac{\varepsilon}{\sqrt{\mu}} \underline{u} \partial_x \underline{v} + \frac{\varepsilon}{\sqrt{\mu}} \frac{1}{h} \partial_x \left(\int_{-1+\beta b}^{\varepsilon \zeta} uv \right) + \frac{1}{\sqrt{\mu} h} \partial_t h \overline{v} + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}} \left(\overline{u} - \underline{u}\right) = 0.$$

Then, thanks to Propositions 2.3, 2.4 and 2.5 we finally obtain that

$$\partial_t \mathrm{Q}_y + \varepsilon \overline{u} \partial_x \mathrm{Q}_y + \varepsilon \mathrm{Q}_x \partial_x \overline{v} + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}} (\overline{u} - \underline{u}) = \mu^{\frac{3}{2}} R.$$

□

Notice that we give in Subsection 4.1 a generalization of the two previous propositions to the fully nonlinear Green-Naghdi regime. Furthermore, in the following proposition we give an equation for $\mathbf{V}^{\sharp}$ up to terms of order $\mathcal{O}\left(\sqrt{\mu}\right)$.

Proposition 2.10. *If* $\left(\zeta, \mathbf{U}_{/\!/}^{\mu,0}, \boldsymbol{\omega}\right)$ *satisfy the Castro-Lannes system* (14)*, then* $\mathbf{V}^{\sharp}$ *satisfies the following equation*

$$\partial_t \mathbf{V}^{\sharp} + \varepsilon \mathbf{V}^{\sharp} \partial_x \overline{u} + \varepsilon \overline{u} \partial_x \mathbf{V}^{\sharp} + \frac{\varepsilon}{\mathrm{Ro}} \mathbf{V}^{\sharp \perp} = \max\left(\varepsilon, \frac{\varepsilon}{\mathrm{Ro}}\right) \sqrt{\mu} R.$$

Proof. The proof is similar to the computation in Part 4.4 in [4]. After multiplying by $(1+z-\beta b)^2$ and integrating with respect to z the second equations of Propositions 2.8 and 2.9, we neglect all the term of order $\mathcal{O}(\sqrt{\mu})$. Then, using the fact that $\partial_t \zeta + \partial_x(h\overline{u}) = 0$ and $\mathbf{V} - \overline{\mathbf{V}} = \sqrt{\mu}\mathbf{V}^*_{\mathrm{sh}} + \mu R$, we get the result. □

2.3 The Boussinesq-Coriolis equations

We can now establish the Boussinesq-Coriolis equations when $d = 1$. The Boussinesq-Coriolis equations are the following system

$$\begin{cases} \partial_t \zeta + \partial_x \left(h\overline{u}\right) = 0, \\ \left(1 - \frac{\mu}{3}\partial_x^2\right)\partial_t \overline{u} + \partial_x \zeta + \varepsilon \overline{u} \partial_x \overline{u} - \frac{\varepsilon}{\mathrm{Ro}}\overline{v} + \frac{\varepsilon}{\mathrm{Ro}}\mu^{\frac{3}{2}}\frac{1}{24}\partial_x^2 \frac{v^\sharp}{h} = 0, \\ \partial_t \overline{v} + \varepsilon \overline{u} \partial_x \overline{v} + \frac{\varepsilon}{\mathrm{Ro}}\overline{u} = 0, \\ \partial_t \mathbf{V}^\sharp + \varepsilon \mathbf{V}^\sharp \partial_x \overline{u} + \varepsilon \overline{u} \partial_x \mathbf{V}^\sharp + \frac{\varepsilon}{\mathrm{Ro}}\mathbf{V}^{\sharp\perp} = 0, \end{cases} \tag{31}$$

where $\mathbf{V}^\sharp$ is defined in (29). We can show that the Boussinesq-Coriolis equations are an order $\mathcal{O}(\mu^2)$ approximation of the water waves equations.

Remark 2.11. *Inspired by [18], we can renormalize* $\mathbf{V}^\sharp$ *by* h *and, using the first equation of* (31)*, we get the following equation*

$$\partial_t \left(\frac{\mathbf{V}^\sharp}{h}\right) + \varepsilon \overline{u} \partial_x \left(\frac{\mathbf{V}^\sharp}{h}\right) + \frac{\varepsilon}{\mathrm{Ro}}\left(\frac{\mathbf{V}^\sharp}{h}\right)^{\perp} = 0.$$

This remark will be useful for the local existence (Proposition 2.15).

Proposition 2.12. *In the Boussinesq regime* $\mathcal{A}_{Bouss}$ (20)*, the Castro-Lannes equations* (14) *are consistent at order* $\mathcal{O}(\mu^2)$ *with the Boussinesq-Coriolis equations* (31) *in the sense of Definition 1.4.*

Proof. The first equation of the Boussinesq-Coriolis equations is always satisfied for a solution of the Castro-Lannes formulation by Proposition 2.3. For the second equation, we use Proposition 2.5, Proposition 2.8 together with Proposition 2.6, Lemma 2.7 and Proposition 2.10 (we recall that $\varepsilon = \mathcal{O}(\mu)$). Notice the fact that all the terms with Q_x disappear. We also use the fact that

$$h^3 v^\sharp = \frac{v^\sharp}{h} + \mu R.$$

Then, the third equation follows from Proposition 2.5, 2.5 and 2.9 (all the terms with Q_y also disappear). □

We notice that contrary to the classical Boussinesq equations, we have a new term due to the vorticity that we can not neglect in presence of a Coriolis forcing. In our knowledge, this term was not highlighted before in the literature.

Remark 2.13. *In the Boussinesq-Coriolis system* (31) *we could simplify the term* $\partial_x^2 \frac{v^\sharp}{h}$ *by* $\partial_x^2 v^\sharp$ *since these terms are equal up to a remainder of order* $\mathcal{O}(\mu)$. *However, the term* $\partial_x^2 \frac{v^\sharp}{h}$ *will be essential for the local existence* (*see Remark 2.16*).

Remark 2.14. *If we assume that* $\frac{\varepsilon}{Ro} = \mathcal{O}\left(\sqrt{\mu}\right)$, *we can neglect the term with* $v^\sharp$ *in the second equation of* (31) *and we obtain*

$$\begin{cases} \partial_t \zeta + \partial_x \left(h \overline{u}\right) = 0, \\ \left(1 - \frac{\mu}{3}\partial_x^2\right) \partial_t \overline{u} + \partial_x \zeta + \varepsilon \overline{u} \partial_x \overline{u} - \frac{\varepsilon}{\mathrm{Ro}} \overline{v} = 0, \\ \partial_t \overline{v} + \varepsilon \overline{u} \partial_x \overline{v} + \frac{\varepsilon}{\mathrm{Ro}} \overline{u} = 0. \end{cases} \tag{32}$$

This system is the classical Boussinesq equations with a standard Coriolis forcing. It is consistent of order $\mathcal{O}(\mu^2)$ *with the Boussinesq-Coriolis equations* (31). *We use this system in Subsections 3.2 and 3.3.*

2.4 Full justification of the Boussinesq-Coriolis equations

In this part, we fully justify the Boussinesq-Coriolis equations (31). In the following we denote by u the quantity $\overline{u}$ and by v the quantity $\overline{v}$. We show that the Boussinesq-Coriolis equations are wellposed. We define the energy space

$$X^s(\mathbb{R}) = H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R}) \times H^{s+1}(\mathbb{R}), \tag{33}$$

endowed with the norm

$$\left|(\zeta, u, v, \mathbf{W})\right|_{X^s_\mu}^2 = |\zeta|_{H^s}^2 + |u|_{H^s}^2 + \mu \left|\partial_x u\right|_{H^s}^2 + |v|_{H^s}^2 + |\mathbf{W}|_{H^s}^2 + \mu \left|\partial_x \mathbf{W}\right|_{H^s}^2. \tag{34}$$

Proposition 2.15. *Let* $A > 0$, $s > \frac{1}{2} + 1$, $\left(\zeta_0, u_0, v_0, \mathbf{V}_0^\sharp\right) \in X^s(\mathbb{R})$ *and* $b \in H^{s+1}(\mathbb{R})$. *We suppose that* $(\varepsilon, \beta, \gamma, \mu, \mathrm{Ro}) \in \mathcal{A}_{Bouss}$. *We assume that*

$$\exists\, h_{\min} > 0 \ , \ \varepsilon \zeta_0 + 1 - \beta b \geq h_{\min}$$

and

$$\left| \left(\zeta_0, u_0, v_0, \frac{\mathbf{V}_0^\sharp}{1 + \varepsilon \zeta_0 - \beta b} \right) \right|_{X^s_\mu} + |b|_{H^{s+1}} \leq A.$$

Then, there exists an existence time $T > 0$ *and a unique solution* $\left(\zeta, u, v, \mathbf{V}^\sharp\right)$ *on* $[0, T]$ *to the Boussinesq-Coriolis equations* (31) *with initial data* $\left(\zeta_0, u_0, v_0, \mathbf{V}_0^\sharp\right)$ *such that we have* $\left(\zeta, u, v, \frac{\mathbf{V}^\sharp}{h}\right) \in \mathcal{C}\left([0, T]; X^s(\mathbb{R})\right)$ *with* $h = 1 + \varepsilon \zeta - \beta b$. *Moreover,*

$$T = \frac{T_0}{\max(\mu, \frac{\varepsilon}{\mathrm{Ro}}\sqrt{\mu})} \ , \ \frac{1}{T_0} = c^1 \text{ and } \max_{[0,T]} \left| \left(\zeta, u, v, \frac{\mathbf{V}^\sharp}{h} \right)(t, \cdot) \right|_{X^s_\mu} = c^2,$$

with $c^j = C\left(A, \mu_{\max}, \frac{1}{h_{\min}}\right)$.

Proof. We only give the energy estimates. For the existence see for instance the proof of Theorem 1 in [14]. We assume that $\left(\zeta, u, v, \mathbf{V}^\sharp\right)$ solves (31) on $\left[0, \frac{T_0}{\max(\mu, \frac{\varepsilon}{\mathrm{Ro}}\sqrt{\mu})}\right]$ and that

$$1 + \varepsilon\zeta - \beta b \geq \frac{h_{\min}}{2} \text{ on } \left[0, \frac{T_0}{\max(\mu, \frac{\varepsilon}{\mathrm{Ro}}\sqrt{\mu})}\right].$$

We denote $U = (\zeta, u, v)^t$ and we focus first on the first three equations. This part is a small adaptation of the proof of Theorem 1 in [14]. The the first three equations of the Boussinesq-Coriolis equations can be symmetrized, as an hyperbolic system, by multiplying the second and the third equations by $h = 1 + \varepsilon\zeta - \beta b$. Then, we obtain the following system

$$\mathcal{A}_0(U)\partial_t U + \mathcal{A}_1(U)\partial_x U + B_1 U + \frac{\varepsilon}{\mathrm{Ro}} B_2(U) U = \frac{\varepsilon}{\mathrm{Ro}} \mu^{\frac{3}{2}} F(h, v^\sharp),$$

where

$$\mathcal{A}_0(U) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & h - \mu\frac{h}{3}\partial_x^2 & 0 \\ 0 & 0 & h \end{pmatrix}, \mathcal{A}_1(U) = \begin{pmatrix} \varepsilon u & h & h \\ h & \varepsilon h u & 0 \\ h & 0 & \varepsilon h u \end{pmatrix}$$

and

$$B_1 = \begin{pmatrix} 0 & -\beta\partial_x b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B_2(U) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -h \\ 0 & h & 0 \end{pmatrix} \text{ and } F(h, v^\sharp) = \begin{pmatrix} 0 \\ -\frac{h}{24}\partial_x^2 \frac{v^\sharp}{h} \\ 0 \end{pmatrix}.$$

Then we remark that $\mathcal{A}_1$ is symmetric and there exists $c_1, c_2 = C\left(\frac{1}{h_{\min}}, |h|_{L^\infty}\right)$ such that

$$c_1 \left|\partial_x f\right|_2^2 \leq \left(-\frac{1}{3}\partial_x\left(h\partial_x f\right), f\right)_2 \leq c_2 \left|\partial_x f\right|_2^2.$$

Hence we introduce the symmetric matrix operator

$$S(U) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & h - \frac{\mu}{3}\partial_x\left(h\partial_x\cdot\right) & 0 \\ 0 & 0 & h \end{pmatrix}$$

and the energy associated

$$\mathcal{E}^s(U) = \left(S(U)\Lambda^s U, \Lambda^s U\right)_2.$$

Then, we see that

$$(\Lambda^s B_2(U)U, \Lambda^s U)_2 = 0$$

and by standard product estimates we get

$$\mu^{\frac{3}{2}} \left| \left(h\Lambda^s \partial_x^2 \frac{v^\sharp}{h}, \Lambda^s u \right)_2 \right| \leq \sqrt{\mu} C(\mathcal{E}^s(U), |b|_{H^{s+1}}) \sqrt{\mu} \left| \frac{v^\sharp}{h} \right|_{H^{s+1}}. \tag{35}$$

Furthermore, notice that

$$\begin{aligned} \mu \left| \partial_t \partial_x u \right|_{H^s} &= \mu \left| \left(1 - \frac{\mu}{3}\partial_x^2\right)^{-1} \partial_x \left(\partial_x \zeta + \varepsilon u \partial_x u - \frac{\varepsilon}{\mathrm{Ro}} v + \frac{\varepsilon}{\mathrm{Ro}} \frac{\mu^{\frac{3}{2}}}{24} \partial_x^2 \frac{v^\sharp}{h} \right) \right|_{H^s}, \\ &\leq C\left(\mu_{\max}, \mathcal{E}^s(U), \sqrt{\mu} \left| \partial_x \frac{v^\sharp}{h} \right|_{H^s} \right). \end{aligned}$$

and therefore

$$\left(\frac{\mu}{3} \partial_x h \Lambda^s \partial_x \partial_t u, \Lambda^s u \right)_2 \leq \mu C\left(\mathcal{E}^s(U), |b|_{H^{s+1}}, \sqrt{\mu} \left| \partial_x \frac{v^\sharp}{h} \right|_{H^s} \right).$$

Gathering all the previous estimate and proceeding as in [14] we obtain

$$\frac{d}{dt} \mathcal{E}^s(U) \leq \max\left(\mu, \frac{\varepsilon}{\mathrm{Ro}} \sqrt{\mu} \right) C\left(\mathcal{E}^s(U), |b|_{H^{s+1}}, \left| \frac{v^\sharp}{h} \right|_{H^s}, \sqrt{\mu} \left| \partial_x \frac{v^\sharp}{h} \right|_{H^s} \right).$$

Furthermore, using Remark 2.11 and the Kato-Ponce estimate, we get

$$\begin{aligned} &\frac{d}{dt} \left| \frac{\mathbf{V}^\sharp}{h} \right|_{H^s}^2 \leq \mu C \left| u \right|_{H^s} \left| \frac{\mathbf{V}^\sharp}{h} \right|_{H^s}^2, \\ &\frac{d}{dt} \mu \left| \partial_x \frac{\mathbf{V}^\sharp}{h} \right|_{H^s}^2 \leq \mu C \left(\sqrt{\mu} \left| \partial_x u \right|_{H^s} \left| \frac{\mathbf{V}^\sharp}{h} \right|_{H^s}^2 + |u|_{H^s} \sqrt{\mu} \left| \partial_x \frac{\mathbf{V}^\sharp}{h} \right|_{H^s} \right) \sqrt{\mu} \left| \partial_x \frac{\mathbf{V}^\sharp}{h} \right|_{H^s}. \end{aligned}$$

Then, the result follows. □

Remark 2.16. *Notice that the previous energy estimates do not imply that $\mathbf{V}^\sharp \in H^{s+1}(\mathbb{R})$. Hence, it is essential that in Inequality* (35) *we have the term $\partial_x^2 \frac{v^\sharp}{h}$ and not simply $\partial_x^2 v^\sharp$ (see Remark 2.13).*

Then, we similarly can prove a local wellposedness result for System (32).

Corollary 2.17. *Let $A > 0$, $s > \frac{1}{2} + 1$, $(\zeta_0, u_0, v_0) \in H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R})$ and $b \in H^{s+1}(\mathbb{R})$. We suppose that $(\varepsilon, \beta, \gamma, \mu, \mathrm{Ro}) \in \mathcal{A}_{Bouss}$. We assume that*

$$\exists\, h_{\min} > 0 \ , \ \varepsilon\zeta_0 + 1 - \beta b \geq h_{\min}$$

and

$$|\zeta_0|_{H^s} + |u_0|_{H^s} + \sqrt{\mu}\,|\partial_x u_0|_{H^s} + |v_0|_{H^s} + |b|_{H^{s+1}} \leq A.$$

Then, there exists an existence time $T > 0$ and a unique solution to the Boussinesq-Coriolis equations (31) $(\zeta, u, v) \in \mathcal{C}\left([0,T]; H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R})\right)$ *with initial data* (ζ_0, u_0, v_0)*. Moreover,*

$$T = \frac{T_0}{\mu}\ ,\ \frac{1}{T_0} = c^1 \text{ and } \max_{[0,T]} |\zeta(t,\cdot)|_{H^s} + |u(t,\cdot)|_{H^s} + \sqrt{\mu}\,|\partial_x u(t,\cdot)|_{H^s} + |v(t,\cdot)|_{H^s} = c^2,$$

with $c^j = C\left(A, \mu_{\max}, \frac{1}{h_{\min}}\right)$.

Furthermore, we have a stability result for the Boussinesq-Coriolis system (31).

Proposition 2.18. *Let the assumptions of Proposition 2.15 satisfied. Suppose that there exists* $\left(\tilde{\zeta}, \tilde{u}, \tilde{v}, \frac{\tilde{\mathbf{V}}^\sharp}{\tilde{h}}\right) \in \mathcal{C}\left(\left[0, \frac{T_0}{\max\left(\mu, \frac{\varepsilon\sqrt{\mu}}{\mathrm{Ro}}\right)}\right]; X^s(\mathbb{R})\right)$ *satisfying*

$$\begin{cases} \partial_t \tilde{\zeta} + \partial_x\left(\tilde{h}\tilde{u}\right) = R_1, \\ \left(1 - \frac{\mu}{3}\partial_x^2\right)\partial_t \tilde{u} + \partial_x \tilde{\zeta} + \varepsilon \tilde{u}\partial_x \tilde{u} - \frac{\varepsilon}{\mathrm{Ro}}\tilde{v} + \frac{\varepsilon}{\mathrm{Ro}}\mu^{\frac{3}{2}}\frac{1}{24}\partial_x \frac{\tilde{v}^\sharp}{\tilde{h}} = R_2, \\ \partial_t \tilde{v} + \varepsilon \tilde{u}\partial_x \tilde{v} + \frac{\varepsilon}{\mathrm{Ro}}\tilde{u} = R_3, \\ \partial_t \frac{\tilde{\mathbf{V}}^\sharp}{\tilde{h}} + \varepsilon\tilde{u}\partial_x \frac{\tilde{\mathbf{V}}^\sharp}{\tilde{h}} + \frac{\varepsilon}{\mathrm{Ro}}\frac{\tilde{\mathbf{V}}^\sharp}{\tilde{h}} = R_4, \end{cases}$$

where $\tilde{h} = 1 + \varepsilon\tilde{\zeta} - \beta b$ *and with* $R = (R_1, R_2, R_3, R_4) \in L^\infty\left(\left[0, \frac{T_0}{\max\left(\mu, \frac{\varepsilon\sqrt{\mu}}{Ro}\right)}\right]; X^s(\mathbb{R})\right)$.

Then, if we denote $\mathfrak{e} = \left(\zeta, u, v, \mathbf{V}^\sharp\right) - \left(\tilde{\zeta}, \tilde{u}, \tilde{v}, \tilde{\mathbf{V}}^\sharp\right)$ *where* $\left(\zeta, u, v, \mathbf{V}^\sharp\right)$ *is the solution given in Proposition 2.15, we have*

$$|\mathfrak{e}(t)|_{X_\mu^{s-1}} \leq C\left(A, \mu_{\max}, \frac{1}{h_{\min}}, \left|\left(\tilde{\zeta}, \tilde{u}, \tilde{v}, \frac{\tilde{\mathbf{V}}^\sharp}{\tilde{h}}, R\right)\right|_{L^\infty\left([0,t]; X_\mu^s \times X_\mu^s\right)}\right)\left(|\mathfrak{e}_{|t=0}|_{X_\mu^{s-1}} + t\,|R|_{X_\mu^s}\right).$$

Proof. This proof is a small adaptation of the one of Proposition 6.5 in [17] (see also [1]). We denote $\tilde{U} = \left(\tilde{\zeta}, \tilde{u}, \tilde{v}\right)$, $\mathfrak{e}_a = U - \tilde{U}$, $R_a = (R_1, R_2, R_3)$ and we keep the notations of the proof of Proposition 2.15. Since the Boussinesq-Coriolis equations are symmetrizable, we have

$$\begin{cases} \mathcal{A}_0(U)\partial_t \mathfrak{e}_a + \mathcal{A}_1(U)\partial_x \mathfrak{e}_a + B_1 \mathfrak{e}_a + \frac{\varepsilon}{\mathrm{Ro}}B_2(U)\mathfrak{e}_a = \frac{\varepsilon}{\mathrm{Ro}}\mu^{\frac{3}{2}}F(h, v^\sharp - \tilde{v}^\sharp) + G, \\ \partial_t\left(\frac{\mathbf{V}^\sharp}{h} - \frac{\tilde{\mathbf{V}}^\sharp}{\tilde{h}}\right) + \varepsilon u \partial_x\left(\frac{\mathbf{V}^\sharp}{h} - \frac{\tilde{\mathbf{V}}^\sharp}{\tilde{h}}\right) + \frac{\varepsilon}{\mathrm{Ro}}\left(\frac{\mathbf{V}^\sharp}{h} - \frac{\tilde{\mathbf{V}}^\sharp}{\tilde{h}}\right)^\perp = H, \end{cases}$$

where

$$G = F(h, \tilde{v}^\sharp) - F(\tilde{h}, \tilde{v}^\sharp) - R_a - (\mathcal{A}_0(U) - \mathcal{A}_0(\tilde{U}))\partial_t \tilde{U} \\ - (\mathcal{A}_1(U) - \mathcal{A}_1(\tilde{U}))\partial_x \tilde{U} - \frac{\varepsilon}{\mathrm{Ro}}(B_2(U) - B_2(\tilde{U}))U,$$

$$H = \varepsilon(\tilde{u} - u)\partial_x \frac{\tilde{\mathbf{V}}^\sharp}{\tilde{h}} + R_4.$$

Then, using standard products estimates, we get (notice that $s > \frac{1}{2} + 1$)

$$\left(\Lambda^{s-1} G, \Lambda^{s-1} \mathfrak{e}_a\right)_2 \leq \left(|R|_{X^s_\mu} + \mu C\left(\mathcal{E}^s(\tilde{U}), \mathcal{E}^{s-1}(\partial_t \tilde{U}), \left|\frac{\tilde{v}^\sharp}{\tilde{h}}\right|_{H^s}, \sqrt{\mu}\left|\partial_x \frac{\tilde{v}^\sharp}{\tilde{h}}\right|_{H^s}\right) |\mathfrak{e}|_{X^{s-1}}\right) |\mathfrak{e}|_{X^{s-1}}$$

and

$$\left(\Lambda^{s-1} H, \Lambda^{s-1}\left(\frac{\mathbf{V}^\sharp}{h} - \frac{\tilde{\mathbf{V}}^\sharp}{\tilde{h}}\right)\right)_2 \leq \left(|R|_{X^s_\mu} + \mu C\left(\mathcal{E}^s(\tilde{U}), \left|\frac{\tilde{\mathbf{V}}^\sharp}{\tilde{h}}\right|_{H^s}, \sqrt{\mu}\left|\partial_x \frac{\tilde{v}^\sharp}{\tilde{h}}\right|_{H^s}\right) |\mathfrak{e}|_{X^{s-1}}\right) |\mathfrak{e}|_{X^{s-1}}.$$

Then, the result follows from energy estimates and the Gronwall's lemma. □

The two previous results and Theorem 1.5 allow us to fully justify the Boussinesq-Coriolis equations. We recall that operators $\overline{\mathbf{V}}[\varepsilon\zeta_0, \beta b](\mathbf{U}^{\mu,0}_{/\!/}, \boldsymbol{\omega})$ and $\mathbf{V}_{\mathrm{sh}}[\varepsilon\zeta, \beta b](\mathbf{U}^{\mu,0}_{/\!/}, \boldsymbol{\omega})(t, X)$ are defined in (22) and (23) respectively.

Theorem 2.19. *Let* $\mathbf{N} \geq 7$ and $(\varepsilon, \beta, \gamma, \mu, \mathrm{Ro}) \in \mathcal{A}_{Bouss}$ *. We assume that we are under the assumptions of Theorem 1.5. Then, we can define the following quantity*

$$(u_0, v_0)^t = \overline{\mathbf{V}}[\varepsilon\zeta_0, \beta b]((\mathbf{U}^{\mu,0}_{/\!/})_0, \boldsymbol{\omega}_0)\ ,\ (u, v)^t = \overline{\mathbf{V}}[\varepsilon\zeta, \beta b](\mathbf{U}^{\mu,0}_{/\!/}, \boldsymbol{\omega}),$$
$$\mathbf{V}^\sharp_0 = \mathbf{V}^\sharp[\varepsilon\zeta_0, \beta b]((\mathbf{U}^{\mu,0}_{/\!/})_0, \boldsymbol{\omega}_0)\ ,\ \mathbf{V}^\sharp = \mathbf{V}^\sharp[\varepsilon\zeta, \beta b](\mathbf{U}^{\mu,0}_{/\!/}, \boldsymbol{\omega}_0),$$

and there exists a time $T > 0$ *such that*

(i) T *has the form*

$$T = \frac{T_0}{\max(\mu, \frac{\varepsilon}{Ro})},\ \text{and}\ \frac{1}{T_0} = c^1.$$

(ii) There exists a unique classical solution $\left(\zeta_B, u_B, v_B, \mathbf{V}^\sharp_B\right)$ *of* (31) *with the initial data* $\left(\zeta_0, u_0, v_0, \mathbf{V}^\sharp_0\right)$ *on* $[0, T]$.

(iii) There exists a unique classical solution $\left(\zeta, \mathbf{U}^{\mu,0}_{/\!/}, \boldsymbol{\omega}\right)$ *of System* (14) *with initial data* $\left(\zeta_0, (\mathbf{U}^{\mu,0}_{/\!/})_0, \boldsymbol{\omega}_0\right)$ *on* $[0, T]$.

(iv) The following error estimate holds, for $0 \leq t \leq T$,

$$\left|\left(\zeta, u, v, \mathbf{V}^{\sharp}\right) - \left(\zeta_B, u_B, v_B, \mathbf{V}_B^{\sharp}\right)\right|_{L^{\infty}([0,t]\times\mathbb{R})} \leq \mu^2 t\, c^2,$$

with $c^j = C\left(A, \mu_{\max}, \frac{1}{h_{\min}}, \frac{1}{\mathfrak{a}_{\min}}, |b|_{H^{N+2}}\right)$.

This theorem shows that the solutions of the water waves system (14) remain close to the solutions of the Boussinesq-Coriolis equations (31) over times $\mathcal{O}\left(\frac{1}{\max(\mu, \frac{\varepsilon}{\mathrm{Ro}})}\right)$ with an accuracy of order $\mathcal{O}(\mu)$. Hence, if one considers a system and wants to show that the solutions of this system remain close to the solutions of the waves equations over times $\mathcal{O}\left(\frac{1}{\max(\mu, \frac{\varepsilon}{\mathrm{Ro}})}\right)$ with an accuracy of order $\mathcal{O}(\mu)$, it is sufficient to compare the solutions of this system with the solutions of the Boussinesq-Coriolis equations (31). It is our approach in the following.

3 Different asymptotic models in the Boussinesq regime over a flat bottom

The Boussinesq-Coriolis equations (31) are particularly interesting for the evolution of offshore water waves. Without vorticity, we get the so-called Boussinesq equations. When we add a rotation, and in particular Coriolis effects, a standard assumption made by physicists is to also assume that the Rossby radius, or Obukhov radius, $\frac{\sqrt{gH}}{f}$ is greater than the typical length of the waves L (see for instance [29], [11], [20]). Then, different regimes for the Coriolis parameter were considered depending on whether the rotation is weak or not ([27], [10], [12]). In this paper, we consider three different regimes (noticed in [10]), a strong rotation ($\frac{\varepsilon}{\mathrm{Ro}} \leq 1$), weak rotation ($\frac{\varepsilon}{\mathrm{Ro}} = \mathcal{O}(\sqrt{\mu})$) and very weak rotation ($\frac{\varepsilon}{\mathrm{Ro}} = \mathcal{O}(\mu)$). We derive and fully justify different asymptotic models when the bottom is flat : a linear equation admitting the so-called Poincaré waves (39) ; the Ostrovsky equation (41), which is a generalization of the KdV equation (50) in presence of a Coriolis forcing, when the rotation is weak; and the KdV equation when the rotation is very weak.

3.1 Strong rotation, the Poincaré waves

In this part we are interested in the behaviour of long water waves under a strong Coriolis forcing (in the sense of [10]). We suppose that $\frac{\varepsilon}{\mathrm{Ro}}$ is of order 1. The asymptotic regime is

$$\mathcal{A}_{\mathrm{Poin}} = \left\{(\varepsilon, \beta, \gamma, \mu, \mathrm{Ro})\,, 0 \leq \mu \leq \mu_0, \varepsilon = \mu, \beta = \gamma = 0, \frac{\varepsilon}{\mathrm{Ro}} = 1\right\}. \tag{36}$$

Then, the Boussinesq-Coriolis equations (31) become

$$\begin{cases} \partial_t \zeta + \partial_x \left((1+\mu\zeta) u \right) = 0, \\ \left(1 - \frac{\mu}{3}\partial_x^2\right) \partial_t u + \partial_x \zeta + \mu u \partial_x u - v + \frac{\mu^{\frac{3}{2}}}{24} \partial_x^2 \frac{v^\sharp}{h} = 0, \\ \partial_t v + \mu u \partial_x v + u = 0, \\ \partial_t \mathbf{V}^\sharp + \mu \mathbf{V}^\sharp \partial_x u + \mu u \partial_x \mathbf{V}^\sharp + \mathbf{V}^{\sharp\perp} = 0. \end{cases} \tag{37}$$

Our purpose is to justify the so-called Poincaré waves or Sverdrup waves ([33]), which are inertia-gravity waves in the linear setting. Dropping all the terms of order $\mathcal{O}(\mu)$ in the Boussinesq-Coriolis equation, we get the linear system

$$\begin{cases} \partial_t \zeta + \partial_x u = 0, \\ \partial_t u + \partial_x \zeta - v = 0, \\ \partial_t v + u = 0. \end{cases} \tag{38}$$

Then, if we denote $U = (\zeta, u, v)^t$, by taking the Fourier transform, we get

$$\partial_t \widehat{U} = \mathcal{A}\widehat{U} \text{ with } \mathcal{A} = \begin{pmatrix} 0 & -i\xi & 0 \\ -i\xi & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

and we obtain,

$$\widehat{U} = \mathcal{S}(t,\xi)\widehat{U}_0 = \begin{pmatrix} \frac{\xi^2 \cos(\sqrt{\xi^2+1}t)+1}{\xi^2+1} & -i\xi \frac{\sin(\sqrt{\xi^2+1}t)}{\sqrt{\xi^2+1}} & i\xi \frac{\cos(\sqrt{\xi^2+1}t)-1}{\xi^2+1} \\ -i\xi \frac{\sin(\sqrt{\xi^2+1}t)}{\sqrt{\xi^2+1}} & \cos(\sqrt{\xi^2+1}t) & \frac{\sin(\sqrt{\xi^2+1}t)}{\sqrt{\xi^2+1}} \\ -i\xi \frac{\cos(\sqrt{\xi^2+1}t)-1}{\xi^2+1} & -\frac{\sin(\sqrt{\xi^2+1}t)}{\sqrt{\xi^2+1}} & \frac{\xi^2+\cos(\sqrt{\xi^2+1}t)}{\xi^2+1} \end{pmatrix} \widehat{U}_0. \tag{39}$$

Commonly, Poincaré waves are waves of the form

$$U(t,x) = e^{i(xk \pm t\sqrt{k^2+1})} U_0.$$

They are solutions of the Klein-Gordon equation. In this setting, Poincaré waves correspond to solutions of System (38) of the form

$$\widehat{U}(t,\xi) = e^{\pm it\sqrt{\xi^2+1}} \widehat{U}_0(\xi).$$

Therefore, a solution of System (38) is a sum of two Poincaré waves if and only if

$$\begin{pmatrix} \frac{1}{\xi^2+1} & 0 & -\frac{i\xi}{\xi^2+1} \\ 0 & 0 & 0 \\ \frac{i\xi}{\xi^2+1} & 0 & \frac{\xi^2}{\xi^2+1} \end{pmatrix} \widehat{U}_0 = 0,$$

which is equivalent to

$$\zeta_0 = \partial_x v_0. \tag{40}$$

In the following, we denote by $\mathcal{S}(t)$ the semi-group of the linear Boussinesq-Coriolis equation. The end of this part is devoted to the full justification of Poincaré waves. The following lemma shows that Condition (40) is propagated by the flow of System (38).

Lemma 3.1. *Let (ζ, u, v) be a solution of (38) such that $(\zeta, u, v)_{|t=0} = 0$ satisfies Condition (40). Then, for all $t \in \mathbb{R}$,*

$$\zeta(t, \cdot) = \partial_x v(t, \cdot).$$

We also have the following dispersion result (see for instance [36] and [25] or Corollary 7.2.4 in [13]).

Lemma 3.2. *Let $u_0 \in W^{2,1}(\mathbb{R})$. Then*

$$\left| \int_{\mathbb{R}} e^{ix\xi \pm t\sqrt{\xi^2+1}} u_0(\xi) d\xi \right|_{L^\infty_x} \leq \frac{C}{\sqrt{1+|t|}} \left| u_0 \right|_{W^{2,1}} .$$

We can give the main result of this part.

Theorem 3.3. *Let $\mu_0 > 0$, $\zeta_0, u_0, v_0, \mathbf{V}_0^\sharp \in H^6(\mathbb{R})$, $x\zeta_0, xu_0, xv_0 \in H^4(\mathbb{R})$, such that ζ_0, v_0 satisfy Condition (40), $1 + \varepsilon\zeta \geq h_{\min} > 0$ and $0 < \mu < \mu_0$. Then, there exists a time $T > 0$, such that there exists*

(i) a unique classical solution $\left(\zeta_B, u_B, v_B, \mathbf{V}_B^\sharp\right)$ of (37) with initial data $\left(\zeta_0, u_0, v_0, \mathbf{V}_0^\sharp\right)$ on $\left[0, \frac{T}{\sqrt{\mu}}\right]$.

(ii) a unique solution (ζ, u, v) of (38) with initial data (ζ_0, u_0, v_0) on $\left[0, \frac{T}{\sqrt{\mu}}\right]$.

Moreover, we have the following error estimate for all $0 \leq t \leq \frac{T}{\sqrt{\mu}}$,

$$\left| (\zeta_B, u_B, v_B) - (\zeta, u, v) \right|_{L^\infty([0,t]\times\mathbb{R})} \leq C \left(\frac{\mu t}{1+\sqrt{t}} + \mu^2 t^2 + \mu^{\frac{3}{2}} t \right) \leq C \mu^{\frac{3}{4}}.$$

where $C = C\left(T, \frac{1}{h_{\min}}, \mu_0, |\zeta_0|_{H^6}, |u_0|_{H^6}, |v_0|_{H^6}, \left|\mathbf{V}_0^\sharp\right|_{H^6}, |x\zeta_0|_{H^4}, |xu_0|_{H^4}, |xv_0|_{H^4}\right)$.

Remark 3.4. *By standard energy estimates, we easily get that, for all $0 \leq t \leq \frac{T}{\sqrt{\mu}}$,*

$$\left| (\zeta_B, u_B, v_B) - (\zeta, u, v) \right|_{L^\infty([0,t]\times\mathbb{R})} \leq C\mu t \leq C\sqrt{\mu},$$

where C is as in the previous theorem. Therefore, our result is not a simple energy estimate. We use the dispersive effects due to the Coriolis forcing to be more accurate.

Proof. The first point follows from Proposition 2.15. For the error estimate, if we denote by $U = (\zeta_B, u_B, v_B)^t$, U satisfies the linear Boussinesq-Coriolis equation up to a remainder of order μ and a remainder of order $\mu^{\frac{3}{2}}$. Then, using the Duhamel's formula we get

$$U(t) = \mathcal{S}(t)U_0 + \mu \int_0^t \mathcal{S}(t-\tau) \begin{pmatrix} -\partial_x \left(\zeta_B u_B\right)(\tau) \\ -u_B(\tau)\partial_x u_B(\tau) + \frac{1}{3}\partial_x^2 \partial_\tau u_B(\tau) \\ -u_B \partial_x v_B \end{pmatrix} + \mu^{\frac{3}{2}} \int_0^t \mathcal{S}(t-\tau)R$$

where R is a remainder bounded uniformly with respect to μ. Then, using again the Duhamel's formula on the first integral we get

$$\begin{aligned} U(t) &= \mathcal{S}(t)U_0 - \mu \int_0^t \mathcal{S}(t-\tau) \begin{pmatrix} \partial_x \left((\mathcal{S}_1(\tau)U_0)(\mathcal{S}_2(\tau)U_0)\right) \\ (\mathcal{S}_2(\tau)U_0)\partial_x(\mathcal{S}_2(\tau)U_0) \\ (\mathcal{S}_2(\tau)U_0)\partial_x(\mathcal{S}_3(\tau)U_0) \end{pmatrix} \\ &\qquad + \mu \int_0^t \mathcal{S}(t-\tau) \begin{pmatrix} 0 \\ \frac{1}{3}\partial_x^2 \partial_\tau \mathcal{S}_2(\tau)U_0 \\ 0 \end{pmatrix} + \mu^2 \int_0^t \int_0^\tau \tilde{R} + \mu^{\frac{3}{2}} \int_0^t \mathcal{S}(t-\tau)\tilde{R} \\ &= \mathcal{S}(t)U_0 - \mu I_1(t) + \mu I_2(t) + \mu^2 I_3(t) + \mu^{\frac{3}{2}} I_4(t), \end{aligned}$$

where $\mathcal{S}_i(t)$ is the ith row of $\mathcal{S}(t)$. We start by estimating I_1. We have

$$I_1(t) = \int_0^t \mathcal{S}(t-\tau) \begin{pmatrix} \partial_x \left(\zeta(\tau)u(\tau)\right) \\ u(\tau)\partial_x u(\tau) \\ u(\tau)\partial_x v(\tau) \end{pmatrix}.$$

Then, we notice that $\partial_x \left(\zeta(\tau)u(\tau)\right) = \partial_x \left(u(\tau)\partial_x v(\tau)\right)$ since $\zeta(\tau) = \partial_x v(\tau)$ by Lemma 3.1. Therefore, using Lemma 3.2 and products estimates, we get

$$\begin{aligned} |I_1(t)|_{L^\infty} &\leq \int_0^t \frac{1}{\sqrt{1+t-\tau}} \left| \begin{pmatrix} \partial_x \left((\mathcal{S}_1(\tau)U_0)(\mathcal{S}_2(\tau)U_0)\right) \\ (\mathcal{S}_2(\tau)U_0)\partial_x(\mathcal{S}_2(\tau)U_0) \\ (\mathcal{S}_2(\tau)U_0)\partial_x(\mathcal{S}_3(\tau)U_0) \end{pmatrix} \right|_{W^{2,1}} \\ &\leq C\left(|\zeta_0|_{H^3}, |u_0|_{H^3}, |v_0|_{H^3}, \left|\mathbf{V}_0^\sharp\right|_{H^3} \right) \frac{t}{\sqrt{1+t}}. \end{aligned}$$

For I_2, using Lemma 3.2 we get

$$|I_2| \leq C\left(|\zeta_0|_{H^4}, |u_0|_{H^4}, |v_0|_{H^4}, |x\zeta_0|_{H^4}, |xu_0|_{H^4}, |xv_0|_{H^4}\right) \frac{t}{\sqrt{1+t}}.$$

Finally, using Proposition 2.15, we have

$$\begin{aligned} |I_3(t)|_{H^1} &\leq C\left(|\zeta_0|_{H^6}, |u_0|_{H^6}, |v_0|_{H^6}, \left|\mathbf{V}_0^\sharp\right|_{H^6} \right) t^2 \\ |I_4(t)|_{H^1} &\leq C\left(|\zeta_0|_{H^4}, |u_0|_{H^4}, |v_0|_{H^4}, \left|\mathbf{V}_0^\sharp\right|_{H^4} \right) t. \end{aligned}$$

Gathering these four estimates, we get the result.

□

Hence, using Theorem 2.19, we justify that poincaré waves remain close to the solutions of the water waves equations (14) over times $\mathcal{O}_{\mu}(1)$ with an accuracy of order $\mathcal{O}(\mu)$. Furthermore, if one can show that a solution of the water waves equations (14), with initial data satisfying Condition (40), exists over a time $\mathcal{O}\left(\frac{1}{\sqrt{\mu}}\right)$, we show that this solution remains close, with an accuracy of order $\mathcal{O}\left(\mu^{\frac{3}{4}}\right)$, to the solution of the linear Boussinesq-Coriolis equations with the same initial data. The reader interested in more linear properties of the water waves equations in shallow water can refer to Chapter 4 in [24].

3.2 Weak rotation, the Ostrovsky equation

Without Coriolis forcing and vorticity, it is well-known, that the KdV equation is a good approximation of the water waves equation under the assumption that ε and μ have the same order ([7], [15], [31], [3], Part 7.1 in [17]). When the Coriolis forcing is taken into account, Ostrovsky ([27]) derived an equation for long waves, which is an adaptation of the KdV equation,

$$\partial_{\xi}\left(\partial_{\tau}k+\frac{3}{2}k\partial_{\xi}k+\frac{1}{6}\partial_{\xi}^{3}k\right)=\frac{1}{2}k. \tag{41}$$

This equation is called the Ostrovsky equation or rKdV-equation in the physical literature. Initially developed for internal water waves, several authors also studied it for surface water waves ([28], [10], [21], [12]). The purpose of this part is to fully justify it. Inspired by [10] we consider the asymptotic regime

$$\mathcal{A}_{\text{Ost}}=\left\{(\varepsilon,\beta,\gamma,\mu,\text{Ro}),0\leq\mu\leq\mu_0,\varepsilon=\mu,\beta=\gamma=0,\frac{\varepsilon}{\text{Ro}}=\sqrt{\mu}\right\}. \tag{42}$$

Then, the Boussinesq-Coriolis equations become (see Remark 2.14)

$$\begin{cases}\partial_t\zeta+\partial_x\left([1+\mu\zeta]u\right)=0,\\ \left(1-\frac{\mu}{3}\partial_x^2\right)\partial_t u+\partial_x\zeta+\mu u\partial_x u-\sqrt{\mu}v=0,\\ \partial_t v+\mu u\partial_x v+\sqrt{\mu}u=0.\end{cases} \tag{43}$$

In order to motivate our approach, let us recall that we are interested in the one-dimensional propagation of water waves in the long wave regime. If we drop all the terms of order $\mathcal{O}(\sqrt{\mu})$ in the Boussinesq-Coriolis, we obtain that

$$\begin{cases}\partial_t\zeta+\partial_x u=0,\\ \partial_t u+\partial_x\zeta=0,\\ \partial_t v=0.\end{cases}$$

Hence, if we assume that v is initially zero, we get a wave equation and propagation of traveling water waves with speed ± 1. Therefore, it is natural to study how these traveling water waves are perturbed when we add weakly nonlinear effects, i.e when we consider the System (43). In this paper, we consider only water waves with speed 1. We seek an approximate solution $(\zeta_{app}, u_{app}, v_{app})$ of (43) under the form

$$\begin{aligned} \zeta_{app}(t,x) &= k(x-t,\mu t) + \mu \zeta_{(1)}(t,x,\mu t), \\ u_{app}(t,x) &= k(x-t,\mu t) + \mu u_{(1)}(t,x,\mu t), \\ v_{app}(t,x) &= \sqrt{\mu} v_{(1/2)}(t,x,\mu t). \end{aligned} \tag{44}$$

where $k = k(\xi, \tau)$ is our modulated traveling water waves, and the others terms are correctors. Then, we plug the ansatz in Sytem (43) and we get

$$\begin{aligned} &\partial_t \zeta_{app} + \partial_x \left([1 + \mu \zeta_{app}] u_{app} \right) = \mu R^1_{(1)} + \mu^2 R_1, \\ &\left(1 - \frac{\mu}{3}\partial_x^2\right) \partial_t u_{app} + \partial_x \zeta_{app} + \mu u_{app} \partial_x u_{app} - \sqrt{\mu} v_{app} = \mu R^2_{(1)} + \mu^2 R_2, \\ &\partial_t v_{app} + \mu u_{app} \partial_x v_{app} + \sqrt{\mu} u_{app} = \sqrt{\mu} R^3_{(1/2)} + \mu^{\frac{3}{2}} R_3, \end{aligned} \tag{45}$$

where

$$\begin{aligned} R^1_{(1)} &= \partial_t \zeta_{(1)} + \partial_x u_{(1)} + \partial_\tau k + 2k\partial_\xi k, \\ R^2_{(1)} &= \partial_t u_{(1)} + \partial_x \zeta_{(1)} + \partial_\tau k + \frac{1}{3}\partial_\xi^3 k + k\partial_\xi k - v_{(1/2)}, \\ R^3_{(1/2)} &= \partial_t v_{(1/2)} + k, \end{aligned}$$

and

$$\begin{aligned} R_1 &= \partial_\tau \zeta_{(1)} + \partial_x \left(k u_{(1)} + k\zeta_{(1)} + \mu \zeta_{(1)} u_{(1)} \right), \\ R_2 &= \partial_\tau u_{(1)} - \frac{1}{3}\partial_\xi^3 \partial_\tau k - \frac{1}{3}\partial_x^2 \partial_t u_{(1)} - \mu \frac{1}{2} \partial_x^3 \partial_\tau u_{(1)} + \partial_x \left(k u_{(1)} \right) + \mu u_{(1)} \partial_x u_{(1)}, \\ R_3 &= \partial_\tau v_{(1/2)} + \left(k + \sqrt{\mu} u_{(1)} \right) \partial_x v_{(1/2)} + u_{(1)}. \end{aligned} \tag{46}$$

Then, the idea is to choose the correctors with $R^1_{(1)}(t,x,\tau) = R^2_{(1)}(t,x,\tau) = 0$ and $R^3_{(1/2)}(t,x,\tau) = 0$ for all $x \in \mathbb{R}$, $t \in \left[0, \frac{T}{\mu}\right]$ and $\tau \in [0, T]$.

Remark 3.5. *In fact, we should add $\sqrt{\mu}\zeta_{(1/2)}(t,x,\mu t)$, $\sqrt{\mu} u_{(1/2)}(t,x,\mu t)$, $v_{(0)}(t,x,\mu t)$, and $\mu v_{(1)}(t,x,\mu t)$ to the ansatz (44) for ζ_{app}, u_{app}, v_{app} and v_{app} respectively. However, if we plug them in System (43) and we want to cancel all the terms of order $\sqrt{\mu}$ and μ, we get*

$$\begin{aligned} &\partial_t \zeta_{(1/2)} + \partial_x u_{(1/2)} = 0, \\ &\partial_t u_{(1/2)} + \partial_x \zeta_{(1/2)} + v_{(0)} = 0, \\ &\partial_t v_{(0)} = 0, \\ &\partial_t v_{(1)} + \partial_\tau v_{(0)} + k\partial_x v_{(0)} + u_{(1/2)} = 0, \end{aligned}$$

which leads to $\zeta_{(1/2)} = u_{(1/2)} = v_{(0)} = v_{(1)} = 0$ *if these quantities are initially zero. Hence, we make this assumption in the following.*

Then, if we assume that $v_{(1/2)}$ and k vanish at $x = \infty$, the condition $R^3_{(1/2)} = 0$ is equivalent to the equation

$$\partial_t \partial_x v_{(1/2)}(t, x, \tau) + \partial_\xi k(x - t, \tau) = 0.$$

Since, $\partial_t(k(x - t, \tau)) = -\partial_\xi k(x - t, \tau)$, we can take

$$\partial_x v_{(1/2)}(t, x, \tau) = \partial_x v^0_{(1/2)}(x) - k^0(x) + k(x - t, \tau), \tag{47}$$

where $v^0_{(1/2)}$ and k^0 are the initial data of $v_{(1/2)}$ and k respectively. Then, we have to introduce the following spaces.

Definition 3.6. *For* $s \in \mathbb{R}$*, we define the Hilbert spaces* $\partial_x H^s(\mathbb{R})$ *as*

$$\partial_x H^s(\mathbb{R}) = \left\{ k \in H^{s-1}(\mathbb{R}),\ k = \partial_x \tilde{k} \text{ with } \tilde{k} \in H^s(\mathbb{R}) \right\},$$

and $\tilde{k}$ *is denoted* $\partial_x^{-1} k$ *in the following. In the same way, we define* $\partial_x^2 H^s(\mathbb{R})$*.*

Then, if we assume that $k(\cdot, \tau) \in \partial_x H^s(\mathbb{R})$ for all $\tau \in [0, T]$, we have

$$v_{(1/2)}(t, x, \tau) = v^0_{(1/2)}(x) - \partial_x^{-1} k^0(x) + \partial_x^{-1} k(x - t, \tau),$$

Furthermore, from $R^1_{(1)} = R^2_{(1)} = 0$, if we denote $w_\pm = \zeta_{(1)} \pm u_{(1)}$ we get

$$\begin{aligned}
&(\partial_t + \partial_x) w_+ + \left(2\partial_\tau k + 3k\partial_\xi k + \frac{1}{3}\partial_\xi^3 k - \partial_\xi^{-1} k \right)(x - t, \tau) - \left(v^0_{(1/2)} - \partial_\xi^{-1} k^0 \right)(x) = 0, \\
&(\partial_t - \partial_x) w_- + \left(k\partial_\xi k - \frac{1}{3}\partial_\xi^3 k + \partial_\xi^{-1} k \right)(x - t, \tau) + \left(v^0_{(1/2)} - \partial_\xi^{-1} k^0 \right)(x) = 0.
\end{aligned} \tag{48}$$

The following lemma (Lemma 7.6 in [17]) is the key point to control u and v.

Lemma 3.7. *Let* $c_1 \neq c_2$*. Let* $k_1, k_2, k_3 \in L^2(\mathbb{R})$ *with* $k_2 = K_2'$ *and* $K_2 \in L^2(\mathbb{R})$*. We consider the unique solution* k *of*

$$\begin{cases} (\partial_t + c_1 \partial_x) k = k_1(x - c_1 t) + k_2(x - c_2 t) + k_3(x - c_2 t), \\ k_{|t=0} = 0. \end{cases}$$

Then, $\lim_{t \to \infty} \left| \frac{1}{t} k(t, \cdot) \right|_2 = 0$ *if and only if* $k_1 \equiv 0$ *and in that case*

$$|k(t, \cdot)|_2 \leq \frac{C}{|c_1 - c_2|} \left(|K_2|_2 \frac{t}{1 + t} + |k_3|_{H^2} \frac{t}{1 + \sqrt{t}} \right).$$

Then, in order to avoid a linear growth for the solution of (48), we also have to impose that

$$\partial_\tau k + \frac{3}{2} k \partial_\xi k + \frac{1}{6}\partial_\xi^3 k = \frac{1}{2}\partial_\xi^{-1} k, \tag{49}$$

which is the Ostrovsky equation. Before giving a full justification of the Ostrovsky equation, we need a local wellposedness result of this equation. The following proposition is a generalization of Theorem 2.1 in [23] and Theorem 2.6 in [35] (see also [22] for weak solutions).

Proposition 3.8. *Let $s > \frac{7}{4}$ and $k_0 \in \partial_x H^s(\mathbb{R})$. Then, there exists a time $T > 0$ and a unique solution $k \in \mathcal{C}\left([0,T]; \partial_x H^s(\mathbb{R})\right)$ to the Ostrovsky equation (49) and one has*

$$\left|\partial_\xi^{-1} k(t,\cdot)\right|_{H^s} \leq C\left(T, \left|\partial_\xi^{-1} k_0\right|_{H^s}\right).$$

Moreover, if $s \geq 3$, $k_0 \in \partial_x^2 H^{s+1}(\mathbb{R})$, $k \in \mathcal{C}\left([0,T]; \partial_x^2 H^{s+1}(\mathbb{R})\right)$ and one has

$$\left|\partial_\xi^{-2} k(t,\cdot)\right|_{H^{s+1}} \leq C\left(T, \left|\partial_\xi^{-2} k_0\right|_{H^{s+1}}\right).$$

Proof. We only prove the second point of the Proposition. We denote by $S(t)$ the semi-group of the linearized Ostrovsky equation

$$\partial_\tau k + \frac{1}{6}\partial_\xi^3 k - \frac{1}{2}\partial_\xi^{-1} k = 0,$$

and it is easy to check that this semi-group acts unitary on $H^s(\mathbb{R})$. We denote $\tilde{k} = \partial_\tau k$. Then, $\tilde{k}$ satisfies the equation

$$\partial_\tau \tilde{k} + \frac{3}{2}\partial_\xi\left(\tilde{k}k\right) + \frac{1}{6}\partial_\xi^3 \tilde{k} - \frac{1}{2}\partial_\xi^{-1}\tilde{k} = 0.$$

Using the Duhamel's formula we obtain

$$\partial_\xi^{-1}\tilde{k}(t,\cdot) = S(t)\partial_\xi^{-1}\tilde{k}_0 + \frac{3}{2}\int_0^t S(t-s)\left(k\tilde{k}\right)(s,\cdot)ds.$$

Since $\partial_\xi^{-1}\tilde{k}_0 = -\frac{3}{4}k_0^2 - \partial_\xi^2 k_0 + \partial_\xi^{-2} k_0 \in L^2(\mathbb{R})$, we get the result since we have

$$\frac{1}{2}\partial_\xi^{-2} k = \partial_\xi^{-1}\partial_\tau k + \frac{3}{4}k^2 + \frac{1}{6}\partial_\xi^2 k.$$

□

Notice that contrary to the KdV equation, we can not expect a global existence. We can now give the main result of this part.

Theorem 3.9. *Let $k^0 \in \partial_x^2 H^{10}(\mathbb{R})$, such that $1+\varepsilon k^0 \geq h_{\min} > 0$, $v^0 \in \partial_x H^6(\mathbb{R})$ and $\mu_0 > 0$. Then, there exists a time $T>0$, such that for all $0 < \mu \leq \mu_0$, we have*

(i) a unique classical solution (ζ_B, u_B, v_B) of (43) with initial data $\left(k^0, k^0, \sqrt{\mu} v^0\right)$ on $\left[0, \frac{T}{\mu}\right]$.

(ii) a unique classical solution k of (49) with initial data k^0 on $[0,T]$.

(iii) If we define $(\zeta_{Ost}, u_{Ostr})(t,x) = (k(x-t,\mu t), k(x-t,\mu t))$ we have the following error estimate for all $0 \leq t \leq \frac{T}{\mu}$,

$$|(\zeta_B, u_B) - (\zeta_{Ost}, u_{Ost})|_{L^\infty([0,t]\times\mathbb{R})} \leq C\left((1+\sqrt{\mu}t)\frac{\mu t}{1+t} + \mu^{\frac{3}{2}} t\right)$$

where $C = C\left(T, \frac{1}{h_{\min}}, \mu_0, \left|\partial_x^{-2} k^0\right|_{H^{10}}, \left|\partial_x^{-1} v^0\right|_{H^6}\right)$.

Proof. In all the proof, C will be a constant as in the theorem. The first and second point follow from Corollary 2.17 and 3.8. In order to get the error estimate, we have to control the remainders R_1, R_2, R_3, defined in (46). First, using Lemma 3.7, the fact that we can express the quantities $\frac{1}{2}\partial_\xi k^2 - \frac{1}{3}\partial_\xi^3 k$, $\partial_\xi^{-1} k$ and v_0 as derivatives with respect to x and the fact that k satisfies the Ostrovsky equation (49), we have

$$\left|\zeta_{(1)}\right|_2 + \left|u_{(1)}\right|_2 \leq C\frac{t}{1+t}.$$

But we can also control all the derivatives with respect to τ or x of u and v be differentiating (48). Hence, we get a control for the remainders R_1 and R_2. For R_3, we use the fact that $v = \partial_x^{-1} k$. We finally, obtain

$$|R_1|_{H^2} + |R_2|_{H^2} + |R_3|_{H^2} \leq C\left(\frac{t}{1+t} + \mu t + 1\right),$$

Then, thanks to Proposition 2.18 and remark 2.14, we get

$$|(\zeta_B, u_B, v_B) - (\zeta_{app}, u_{app}, v_{app})|_{L^\infty([0,t]\times\mathbb{R})} \leq C\mu^{\frac{3}{2}} t\left(\frac{t}{1+t} + \mu t + 1\right).$$

Moreover, we have

$$|(\zeta_{app}, u_{app}) - (\zeta_{Ost}, v_{Ost})|_{L^\infty([0,t]\times\mathbb{R})} \leq \mu\frac{t}{1+t}.$$

Then, the result follows easily.

□

This theorem, combined with Theorem 2.19, shows that the solutions of the water waves equations (14) is well approximated over times $\mathcal{O}\left(\frac{1}{\sqrt{\mu}}\right)$ with an accuracy of order $\mathcal{O}(\mu)$ by the Ostrovsky approximation if we have a small Coriolis forcing. The approach we develop here is similar to the one of the KP equations (see for instance [19], [1] or Part

7.2.1 in [17]). The fact that $k^0 \in \partial_x H^8$ is essential and physical since a solution of the Ostrovsky equation has to be mean free. However, we suppose here that $k^0 \in \partial_x^2 H^9(\mathbb{R})$ and $v^0 \in \partial_x H^5(\mathbb{R})$ which is more restrictive. In fact, using the strategy developed in [2] for the KP approximation we can hope to release this assumption. Finally, notice that contrary to the KdV equation, the Ostrovsky equation does not admit solitons ([38], [9]).

3.3 Very weak rotation, the KdV equation

As we said before, without Coriolis forcing, it is well-known, that the KdV equation is a good approximation of the water waves equations. In this part we show that if $\frac{\varepsilon}{\mathrm{Ro}}$ is small enough, we get the KdV equation as an asymptotic model. We recall the KdV equation

$$\partial_\tau k + \frac{3}{2} k \partial_\xi k + \frac{1}{6} \partial_\xi^3 k = 0. \tag{50}$$

Inspired by [10], we show that $\frac{\varepsilon}{\mathrm{Ro}} = \mathcal{O}(\mu)$ is sufficient. we consider the asymptotic regime

$$\mathcal{A}_{KdV} = \left\{ (\varepsilon, \beta, \gamma, \mu, \mathrm{Ro}), 0 \leq \mu \leq \mu_0, \varepsilon = \mu, \beta = \gamma = 0, \frac{\varepsilon}{\mathrm{Ro}} = \mu \right\}. \tag{51}$$

Then, the Boussinesq-Coriolis equations become (see Remark 2.14)

$$\begin{cases} \partial_t \zeta + \partial_x \left([1 + \mu\zeta] u \right) = 0, \\ \left(1 - \frac{\mu}{3} \partial_x^2 \right) \partial_t u + \partial_x \zeta + \mu u \partial_x u - \mu v = 0, \\ \partial_t v + \mu u \partial_x v + \mu u = 0. \end{cases} \tag{52}$$

Proceeding as in the previous part, we seek an approximate solution $(\zeta_{app}, u_{app}, v_{app})$ of (52) under the form

$$\begin{aligned} \zeta_{app}(t,x) &= k(x-t, \mu t) + \mu \zeta_{(1)}(t, x, \mu t), \\ u_{app}(t,x) &= k(x-t, \mu t) + \mu u_{(1)}(t, x, \mu t), \\ v_{app}(t,x) &= \mu v_{(1)}(t, x, \mu t). \end{aligned} \tag{53}$$

Then, we plug the ansatz in Sytem (52) and we get

$$\begin{aligned} &\partial_t \zeta_{app} + \partial_x \left([1 + \mu\zeta_{app}] u_{app} \right) = \mu R^1_{(1)} + \mu^2 R_1, \\ &\left(1 - \frac{\mu}{3} \partial_x^2 \right) \partial_t u_{app} + \partial_x \zeta_{app} + \mu u_{app} \partial_x u_{app} - \mu v_{app} = \mu R^2_{(1)} + \mu^2 R_2, \\ &\partial_t v_{app} + \mu u_{app} \partial_x v_{app} + \mu u_{app} = \mu R^3_{(1)} + \mu^2 R_3, \end{aligned} \tag{54}$$

where

$$
\begin{aligned}
R^1_{(1)} &= \partial_t \zeta_{(1)} + \partial_x u_{(1)} + \partial_\tau k + 2k\partial_\xi k, \\
R^2_{(1)} &= \partial_t u_{(1)} + \partial_x \zeta_{(1)} + \partial_\tau k + \frac{1}{3}\partial_\xi^3 k + k\partial_\xi k, \\
R^3_{(1)} &= \partial_t v_{(1)} + k,
\end{aligned}
$$

and

$$
\begin{aligned}
R_1 &= \partial_\tau \zeta_{(1)} + \partial_x \left(k u_{(1)} + k\zeta_{(1)} + \mu \zeta_{(1)} u_{(1)} \right), \\
R_2 &= \partial_\tau u_{(1)} - \frac{1}{3}\partial_\xi^3 \partial_\tau k - \frac{1}{3}\partial_x^2 \partial_t u_{(1)} - \mu \frac{1}{3}\partial_x^2 \partial_\tau u_{(1)} + \partial_x \left(k u_{(1)} \right) + \mu u_{(1)} \partial_x u_{(1)} - v_{(1)}, \\
R_3 &= \partial_\tau v_{(1)} + \mu \left(k + \mu u_{(1)} \right) \partial_x v_{(1)} + u_{(1)}.
\end{aligned}
$$

Remark 3.10. *We should also add $v_{(0)}(t, x, \mu t)$ to the ansatz (53) for v_{app}. However, if we plug it in System (52) we get $\partial_t v_{(0)} = 0$ which leads to $v_{(0)} = 0$ if the quantity is initially zero. Hence, we make this assumption in the following.*

As before, we assume that $R^1_{(1)}(t, x, \tau) = R^2_{(1)}(t, x, \tau) = R^3_{(1)}(t, x, \tau) = 0$ for all $x \in \mathbb{R}$, $t \in \left[0, \frac{T}{\mu}\right]$ and $\tau \in [0, T]$ which leads to $v_{(1)} = v^0_{(1)} - \partial_x^{-1} k^0 + \partial_x^{-1} k$ and, if we denote $w_\pm = \zeta_{(1)} \pm u_{(1)}$ we get

$$
\begin{aligned}
&\left(\partial_t + \partial_x\right) w_+ + \left(2\partial_\tau k + 3k\partial_\xi k + \frac{1}{3}\partial_\xi^3 k \right)(x - t, \tau) = 0, \\
&\left(\partial_t - \partial_x\right) w_- + \left(k\partial_\xi k - \frac{1}{3}\partial_\xi^3 k \right)(x - t, \tau) = 0
\end{aligned}
$$

and to avoid a linear growth of u or v we need that k satisfies (50). We also have an existence result for the KdV equation (see for instance [16]).

Proposition 3.11. *Let $s \geq 1$, $k_0 \in H^s(\mathbb{R})$ and $T > 0$. Then, there exists a unique solution to the KdV equation (50) $k \in \mathcal{C}\left([0, T]; H^s(\mathbb{R})\right)$ and one have*

$$
|k|_{H^s} \leq C\left(T, |k_0|_{H^s}\right).
$$

Moreover, if $s \geq 2$ and $k_0 \in \partial_x H^{s+1}(\mathbb{R})$, $k \in \mathcal{C}\left([0, T]; \partial_x H^{s+1}(\mathbb{R})\right)$ and we have

$$
\left|\partial_x^{-1} k\right|_{H^{s+1}} \leq C\left(T, \left|\partial_x^{-1} k_0\right|_{H^{s+1}}\right).
$$

Then, proceeding as in the previous part, we obtain the following theorem.

Theorem 3.12. *Let $k^0 \in \partial_x H^9(\mathbb{R})$, such that such that $1 + \varepsilon k^0 \geq h_{\min} > 0$, $v^0 \in H^5(\mathbb{R})$ and $\mu_0 > 0$. Then, there exists a time $T > 0$, such that for all $0 < \mu \leq \mu_0$, we have*

(i) a unique classical solution (ζ_B, u_B, v_B) of (52) with initial data $\left(k^0, k^0, \mu v^0\right)$ on $\left[0, \frac{T}{\mu}\right]$.

(ii) a unique classical solution k of (50) *with initial data k^0 on* $[0,T]$.

(iii) If we define $(\zeta_{KdV}, u_{KdV})(t,x) = (k(x-t,\mu t), k(x-t,\mu t))$ *we have the following error estimate for all* $0 \leq t \leq \frac{T}{\mu}$,

$$\left|(\zeta_B, u_B) - (\zeta_{KdV}, u_{KdV})\right|_{L^\infty([0,t]\times\mathbb{R})} \leq C\left(\frac{\mu t}{1+t} + \mu^2 t\right)$$

where $C = C\left(T, \frac{1}{h_{\min}}, \mu_0, \left|\partial_x^{-1} k^0\right|_{H^9}, \left|v^0\right|_{H^5}\right)$.

This theorem, combined with Theorem 2.19, shows that the solutions of the water waves equations (14) is well approximated over times $\mathcal{O}\left(\frac{1}{\mu}\right)$ with an accuracy of order $\mathcal{O}(\mu)$ by the KdV approximation if we have a very small Coriolis forcing. Notice that contrary to the irrotational case, the transverse velocity v is not zero (also noticed in [10]). Furthermore, in our situation, the initial data for the KdV equation has to be of zero mean which means that we can not expect the propagation of solitons on a large time (they have a constant sign) if $\frac{\varepsilon}{\mathrm{Ro}}$ and μ have the same order.

4 Green-Naghdi equations for $\gamma = 0$ and $\beta = \mathcal{O}(\mu)$

This part is devoted to the derivation and justification of the Green-Naghdi equations (62) under a Coriolis forcing, with $\gamma = 0$ and for small amplitude topography variations ($\beta = \mathcal{O}(\mu)$). The Green-Naghdi equations are originally obtained in the irrotational framework under the assumption that μ is small (no assumption on ε) and by neglecting all the terms of order $\mathcal{O}(\mu^2)$ in the water waves equations (see for instance [32] or Part 5.1.1.2 in [17]). It is a system of two equations on the surface ζ and the averaged horizontal velocity $\overline{\mathbf{V}}$. These equations were generalized in [4] in presence of vorticity but without a Coriolis forcing. This new system is a cascade of equations that involves a second order tensor and a third order tensor. After deriving these equations, we show that they are an order $\mathcal{O}(\mu^2)$ approximation of the water waves equations. We consider the asymptotic regime for the 1D Green-Naghdi equations

$$\mathcal{A}_{\mathrm{GN}} = \left\{(\varepsilon, \beta, \gamma, \mu, \mathrm{Ro}), 0 \leq \mu \leq \mu_0, 0 \leq \varepsilon, \frac{\varepsilon}{\mathrm{Ro}} \leq 1, \beta = \mathcal{O}(\mu), \gamma = 0\right\}. \tag{55}$$

The next subsection is devoted to extending Proposition 2.8 and 2.9.

4.1 Improvements for the equations of $\mathbf{Q}_x$ and $\mathbf{Q}_y$

We start by extending Proposition 2.8.

Proposition 4.1. *If* $\left(\zeta, \mathbf{U}_{/\!/}^{\mu,0}, \boldsymbol{\omega}\right)$ *satisfy the Castro-Lannes system* (14)*, then Q_x satisfies the following equation*

$$\begin{aligned}\partial_t Q_x + \varepsilon\overline{u}\partial_x Q_x + \varepsilon Q_x\partial_x\overline{u} + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}}\left(\underline{v} - \overline{v}\right) = & -\varepsilon\sqrt{\mu}\frac{1}{h}\partial_x\int_{-1+\beta b}^{\varepsilon\zeta}\left(u_{\mathrm{sh}}^*\right)^2 \\ & + \varepsilon\sqrt{\mu}\mathrm{Q}_x\partial_x\mathrm{Q}_x + \varepsilon\mu\frac{1}{3}\partial_x\left(h^2\mathrm{Q}_x\partial_x^2\overline{u}\right) \\ & + \varepsilon\mu\frac{1}{6}h^2u^\sharp\partial_x^3\overline{u} + \varepsilon\mu\frac{1}{8h}\partial_x\left(h^3u^\sharp\right)\partial_x^2\overline{u} \\ & + \varepsilon\max\left(\beta\sqrt{\mu},\mu^{\frac{3}{2}}\right)R,\end{aligned}$$

and u_{sh}^* satisfies the equation

$$\begin{aligned}\partial_t u_{\mathrm{sh}}^* + \varepsilon\overline{u}\partial_x u_{\mathrm{sh}}^* + \varepsilon u_{\mathrm{sh}}^*\partial_x\overline{u} + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}}\left(\overline{v} - v\right) = & \varepsilon\sqrt{\mu}\frac{1}{h}\partial_x\int_{-1+\beta b}^{\varepsilon\zeta}\left(u_{\mathrm{sh}}^*\right)^2 - \varepsilon\sqrt{\mu}u_{\mathrm{sh}}^*\partial_x u_{\mathrm{sh}}^* \\ & + \varepsilon\partial_x\left(\int_{-1+\beta b}^{z}\left[\overline{u} + \sqrt{\mu}u_{\mathrm{sh}}^*\right]\right)\partial_z u_{\mathrm{sh}}^* \\ & + \varepsilon\mu R.\end{aligned}$$

Proof. Using the second equation of the vorticity equation of the Castro-Lannes system (14), we have

$$\partial_t\boldsymbol{\omega}_y + \varepsilon u\partial_x\boldsymbol{\omega}_y + \frac{\varepsilon}{\mu}\mathrm{w}\partial_z\boldsymbol{\omega}_y = \varepsilon\boldsymbol{\omega}_x\partial_x v + \frac{\varepsilon}{\sqrt{\mu}}\boldsymbol{\omega}_z\partial_z v + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}}\partial_z v.$$

Since $\boldsymbol{\omega}_x = -\frac{1}{\sqrt{\mu}}\partial_z v$ and $\boldsymbol{\omega}_z = \partial_x v$ we notice that $\varepsilon\boldsymbol{\omega}_x\partial_x v + \frac{\varepsilon}{\sqrt{\mu}}\boldsymbol{\omega}_z\partial_z v = 0$. Using Proposition 2.5 we get

$$\partial_t\boldsymbol{\omega}_y + \varepsilon\overline{u}\partial_x\boldsymbol{\omega}_y - \varepsilon\partial_x\left[(1+z-\beta b)\overline{u}\right]\partial_z\boldsymbol{\omega}_y - \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}}\partial_z v + \varepsilon\sqrt{\mu}A_1 + \varepsilon\mu A_2 = \varepsilon\max\left(\mu^{\frac{3}{2}},\beta\sqrt{\mu}\right)R,$$

where

$$\begin{aligned}A_1 &= u_{\mathrm{sh}}^*\partial_x\boldsymbol{\omega}_y - \partial_x\left(\int_{-1+\beta b}^{z}u_{\mathrm{sh}}^*\right)\partial_z\boldsymbol{\omega}_y, \\ A_2 &= -\frac{1}{2}\left([1+z-\beta b]^2 - \frac{h^2}{3}\right)\partial_x^2\overline{u}\partial_x\boldsymbol{\omega}_y + \frac{1}{2}\partial_x\left(\int_{-1+\beta b}^{z}\left([1+z-\beta b]^2 - \frac{h^2}{3}\right)\partial_x^2\overline{u}\right)\partial_z\boldsymbol{\omega}_y.\end{aligned}$$

Then, integrating with respect to z, using the fact that $\partial_t\zeta + \partial_x(h\overline{u}) = 0$ and $u_{\mathrm{sh}} = -\int_z^{\varepsilon\zeta}\boldsymbol{\omega}_y$, we get

$$\begin{aligned}\partial_t u_{\mathrm{sh}} + \varepsilon\overline{u}\partial_x u_{\mathrm{sh}} + \varepsilon u_{\mathrm{sh}}\partial_x\overline{u} + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}}\left(\underline{v} - v\right) = & \varepsilon\partial_x\left[(1+z-\beta b)\,\overline{u}\right]\partial_z u_{\mathrm{sh}} + \varepsilon\sqrt{\mu}\int_z^{\varepsilon\zeta}A_1 \\ & + \varepsilon\mu\int_z^{\varepsilon\zeta}A_2 + \varepsilon\max\left(\mu^{\frac{3}{2}},\beta\sqrt{\mu}\right)R.\end{aligned}$$

Integrating again with respect to z, using the fact that $\partial_t \zeta + \partial_x (h\overline{u}) = 0$ and $Q_x = \overline{u_{\mathrm{sh}}^*}$, we obtain

$$\partial_t Q_x + \varepsilon \overline{u} \partial_x Q_x + \varepsilon Q_x \partial_x \overline{u} + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}} \left(\underline{v} - \overline{v} \right) = \varepsilon \sqrt{\mu} \frac{1}{h} \int_{-1+\beta b}^{\varepsilon\zeta} \int_{z}^{\varepsilon\zeta} A_1 + \varepsilon \mu \frac{1}{h} \int_{-1+\beta b}^{\varepsilon\zeta} \int_{z}^{\varepsilon\zeta} A_2 + \varepsilon \max\left(\mu^{\frac{3}{2}}, \beta\sqrt{\mu} \right) R.$$

The end of the proof is devoted to the computation of the others terms. We have

$$\begin{aligned}\int_{z}^{\varepsilon\zeta} A_1 &= \int_{z}^{\varepsilon\zeta} u_{\mathrm{sh}}^* \partial_x \boldsymbol{\omega}_y - \partial_x \left(\int_{-1+\beta b}^{z} u_{\mathrm{sh}}^* \right) \partial_z \boldsymbol{\omega}_y \\ &= \int_{z}^{\varepsilon\zeta} \partial_x \left(u_{\mathrm{sh}}^* \boldsymbol{\omega}_y \right) - \varepsilon \zeta \mathrm{Q}_x \underline{\boldsymbol{\omega}_y} + \partial_x \left(\int_{-1+\beta b}^{z} u_{\mathrm{sh}}^* \right) \boldsymbol{\omega}_y.\end{aligned}$$

Since $\boldsymbol{\omega}_y = \partial_z u_{\mathrm{sh}}^*$, we obtain

$$\int_{z}^{\varepsilon\zeta} A_1 = \mathrm{Q}_x \partial_x \mathrm{Q}_x - u_{\mathrm{sh}}^* \partial_x u_{\mathrm{sh}}^* + \partial_x \left(\int_{-1+\beta b}^{z} u_{\mathrm{sh}}^* \right) \partial_z u_{\mathrm{sh}}^*.$$

then, integrating again with respect to z, we obtain

$$\frac{1}{h} \int_{-1+\beta b}^{\varepsilon\zeta} \int_{z}^{\varepsilon\zeta} A_1 = \mathrm{Q}_x \partial_x \mathrm{Q}_x - \frac{1}{h} \partial_x \int_{-1+\beta b}^{\varepsilon\zeta} \left(u_{\mathrm{sh}}^* \right)^2.$$

Furthermore, we have

$$\begin{aligned}\int_{z}^{\varepsilon\zeta} A_2 &= -\frac{1}{2} \int_{z}^{\varepsilon\zeta} \left(\left[1 + z' - \beta b \right]^2 - \frac{h^2}{3} \right) \partial_x^2 \overline{u} \partial_x \boldsymbol{\omega}_y \\ &\quad + \frac{1}{2} \int_{z}^{\varepsilon\zeta} \partial_x \left(\int_{-1+\beta b}^{z} \left(\left[1 + z' - \beta b \right]^2 - \frac{h^2}{3} \right) \partial_x^2 \overline{u} \right) \partial_z \boldsymbol{\omega}_y \\ &= -\frac{1}{2} \int_{z}^{\varepsilon\zeta} \partial_x \left[\left(\left[1 + z' - \beta b \right]^2 - \frac{h^2}{3} \right) \partial_x^2 \overline{u} \boldsymbol{\omega}_y \right] - \varepsilon \partial_x \zeta \frac{h^2}{3} \partial_x^2 \overline{u} \underline{\boldsymbol{\omega}_y} \\ &\quad - \frac{1}{2} \partial_x \left(\int_{-1+\beta b}^{z} \left(\left[1 + z' - \beta b \right]^2 - \frac{h^2}{3} \right) \partial_x^2 \overline{u} \right) \boldsymbol{\omega}_y.\end{aligned}$$

Since $\boldsymbol{\omega}_y = \partial_z u_{\mathrm{sh}}^*$, we obtain

$$\begin{aligned}\int_{z}^{\varepsilon\zeta} A_2 &= \int_{z}^{\varepsilon\zeta} \partial_x \left(\left[1 + z' - \beta b \right] \partial_x^2 \overline{u} u_{\mathrm{sh}}^* \right) + \frac{1}{2} \partial_x \left(\left(\left[1 + z' - \beta b \right]^2 - \frac{h^2}{3} \right) \partial_x^2 \overline{u} u_{\mathrm{sh}}^* \right) \\ &\quad - \frac{1}{2} \partial_x \left(\int_{-1+\beta b}^{z} \left(\left[1 + z' - \beta b \right]^2 - \frac{h^2}{3} \right) \partial_x^2 \overline{u} \right) \partial_z u_{\mathrm{sh}}^* \\ &\quad + \frac{1}{3} \partial_x \left(h^2 \partial_x^2 \overline{u} \mathrm{Q}_x \right) - \varepsilon \partial_x \zeta h \partial_x^2 \overline{u} \mathrm{Q}_x.\end{aligned}$$

Then we integrate again with respect to z and we divide h. We obtain

$$
\begin{aligned}
\frac{1}{h}\int_{-1+\beta b}^{\varepsilon\zeta}\int_{z}^{\varepsilon\zeta} A_2 =& \frac{1}{h}\int_{-1+\beta b}^{\varepsilon\zeta}\int_{z}^{\varepsilon\zeta} \partial_x \left(\left[1+z'-\beta b\right] \partial_x^2 \overline{u} u_{\mathrm{sh}}^* \right) \\
&+ \frac{1}{2h}\int_{-1+\beta b}^{\varepsilon\zeta} \partial_x \left(\left(\left[1+z'-\beta b\right]^2 - \frac{h^2}{3}\right) \partial_x^2 \overline{u} u_{\mathrm{sh}}^* \right) \\
&+ \frac{1}{2h}\int_{-1+\beta b}^{\varepsilon\zeta} \partial_x \left(\left(\left[1+z'-\beta b\right]^2 - \frac{h^2}{3}\right) \partial_x^2 \overline{u} \right) u_{\mathrm{sh}}^* \\
&+ \frac{1}{3}\partial_x \left(h^2 \partial_x^2 \overline{u} \mathrm{Q}_x \right) - \frac{4}{3} h \partial_x h \partial_x^2 \overline{u} \mathrm{Q}_x + \beta R.
\end{aligned}
$$

Then, using the fact that

$$
\int_{-1+\beta b}^{\varepsilon\zeta}\int_{z}^{\varepsilon\zeta}\int_{-1+\beta b}^{z'} \partial_x u_{\mathrm{sh}}^* = \partial_x \int_{-1+\beta b}^{\varepsilon\zeta}\int_{z}^{\varepsilon\zeta}\int_{-1+\beta b}^{z'} u_{\mathrm{sh}}^* + \beta R,
$$

we finally get

$$
\frac{1}{h}\int_{-1+\beta b}^{\varepsilon\zeta}\int_{z}^{\varepsilon\zeta} A_2 = \frac{1}{3}\partial_x \left(h^2 \mathrm{Q}_x \partial_x^2 \overline{u} \right) + \frac{1}{6} h^2 u^\sharp \partial_x^3 \overline{u} + \frac{1}{8h}\partial_x \left(h^3 u^\sharp \right) \partial_x^2 \overline{u} + \beta R,
$$

and the first equation follows. The second equation follows similarly using the fact that $u_{\mathrm{sh}}^* = u_{\mathrm{sh}} - \mathrm{Q}_x$. □

We can also extend Proposition 2.9.

Proposition 4.2. *If* $\left(\zeta, \mathbf{U}_{/\!/}^{\mu,0}, \boldsymbol{\omega}\right)$ *satisfy the Castro-Lannes system* (14)*, then* Q_x *satisfies the following equation*

$$
\begin{aligned}
\partial_t \mathrm{Q}_y + \varepsilon \overline{u} \partial_x \mathrm{Q}_y + \varepsilon \mathrm{Q}_x \partial_x \overline{v} + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}} \left(\overline{u} - \underline{u} \right) =& \varepsilon \sqrt{\mu} \mathrm{Q}_x \partial_x \mathrm{Q}_y - \varepsilon \sqrt{\mu} \frac{1}{3} h^2 \partial_x^2 \overline{u} \partial_x \overline{v} \\
&- \varepsilon \sqrt{\mu} \frac{1}{h} \partial_x \left(\int_{-1+\beta b}^{\varepsilon\zeta} u_{\mathrm{sh}}^* v_{\mathrm{sh}}^* \right) \\
&- \varepsilon \mu \left(\partial_x h \right)^2 \mathrm{Q}_x \partial_x \overline{v} + \varepsilon \mu \frac{h^2}{3} \partial_x^2 \overline{u} \partial_x \mathrm{Q}_y \\
&- \varepsilon \mu \frac{1}{24h} \partial_x^2 \left(h^3 u^\sharp \right) \partial_x \overline{v} + \varepsilon \mu \frac{1}{24h} \partial_x \left(h^3 v^\sharp \partial_x^2 \overline{u} \right) \\
&+ \varepsilon \max \left(\mu^{\frac{3}{2}}, \beta \sqrt{\mu} \right) R,
\end{aligned}
$$

and v_{sh}^* *satisfies the equation*

$$
\begin{aligned}
\partial_t v^*_{\mathrm{sh}} + \varepsilon \overline{u} \partial_x v^*_{\mathrm{sh}} + \varepsilon u^*_{\mathrm{sh}} \partial_x \overline{v} + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}} (u - \overline{u}) =& \varepsilon\sqrt{\mu}\frac{1}{h}\partial_x \left(\int_{-1+\beta b}^{\varepsilon\zeta} u^*_{\mathrm{sh}} v^*_{\mathrm{sh}} \right) - \varepsilon\sqrt{\mu} u^*_{\mathrm{sh}} \partial_x v^*_{\mathrm{sh}} \\
&+ \varepsilon \partial_x \left(\int_{-1+\beta b}^{z} [\overline{u} + \sqrt{\mu} u^*_{\mathrm{sh}}] \right) \partial_z v^*_{\mathrm{sh}} \\
&+ \varepsilon\sqrt{\mu}\frac{1}{2} \left([1 + z - \beta b]^2 - \frac{h^2}{3} \right) \partial_x^2 \overline{u} \partial_x \overline{v} \\
&+ \varepsilon \left(\mu, \beta\sqrt{\mu}\right) R.
\end{aligned}
$$

Proof. Using the first equation of the vorticity equation of the Castro-Lannes system (14), we have

$$
\partial_t \boldsymbol{\omega}_x + \varepsilon u \partial_x \boldsymbol{\omega}_x + \frac{\varepsilon}{\mu} \mathrm{w} \partial_z \boldsymbol{\omega}_x = \varepsilon \boldsymbol{\omega}_x \partial_x u + \frac{\varepsilon}{\sqrt{\mu}} \boldsymbol{\omega}_z \partial_z u + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}} \partial_z u.
$$

Then, using the fact that $\nabla^{\mu,0} \cdot \boldsymbol{\omega} = 0$ and $\nabla^{\mu,0} \cdot \mathbf{U}^{\mu,\gamma} = 0$, we get

$$
\partial_t \boldsymbol{\omega}_x - \frac{\varepsilon}{\sqrt{\mu}} \partial_z \left(u \boldsymbol{\omega}_z\right) + \frac{\varepsilon}{\mu} \partial_z \left(\mathrm{w} \boldsymbol{\omega}_x\right) = \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}} \partial_z u.
$$

then, we integrate with respect to z and, using the fact that $\partial_t \zeta - \frac{1}{\mu} \underline{\mathbf{U}}^{\mu} \cdot \mathbf{N}^{\mu,0} = 0$, $\boldsymbol{\omega}_x = -\frac{1}{\sqrt{\mu}} \partial_z v$ and $\boldsymbol{\omega}_z = \partial_x v$, we obtain

$$
\partial_t \left(\int_{-1+\beta b}^{\varepsilon\zeta} \boldsymbol{\omega}_x \right) - \frac{\varepsilon}{\sqrt{\mu}} \underline{u} \partial_x \underline{v} + \frac{\varepsilon}{\sqrt{\mu}} u \partial_x v + \frac{\varepsilon}{\mu^{\frac{3}{2}}} \mathrm{w} \partial_z v + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}} (u - \underline{u}) = 0.
$$

Then, we integrate again with respect to z and, using Proposition 2.4 and the fact that $\partial_t \zeta - \frac{1}{\mu} \underline{\mathbf{U}}^{\mu} \cdot \mathbf{N}^{\mu,0} = 0$, $\mathbf{U}^{\mu}_b \cdot \mathbf{N}^{\mu,0}_b = 0$, and $\nabla^{\mu,0} \cdot \mathbf{U}^{\mu} = 0$, we get

$$
\partial_t \mathrm{Q}_y - \frac{\varepsilon}{\sqrt{\mu}} \underline{u} \partial_x \underline{v} + \frac{\varepsilon}{\sqrt{\mu}} \frac{1}{h} \partial_x \left(\int_{-1+\beta b}^{\varepsilon\zeta} uv \right) + \frac{1}{\sqrt{\mu} h} \partial_t h \overline{v} + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}} \left(\overline{u} - \underline{u}\right) = 0.
$$

Then, thanks to Propositions 2.3, 2.4 and 2.5 we finally obtain that

$$
\begin{aligned}
\partial_t \mathrm{Q}_y + \varepsilon \overline{u} \partial_x \mathrm{Q}_y + \varepsilon \mathrm{Q}_x \partial_x \overline{v} + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}} (\overline{u} - \underline{u}) =& \varepsilon\sqrt{\mu} \mathrm{Q}_x \partial_x \mathrm{Q}_y - \varepsilon\sqrt{\mu} \frac{1}{3} h^2 \partial_x^2 \overline{u} \partial_x \overline{v} \\
&- \varepsilon\sqrt{\mu}\frac{1}{h}\partial_x \left(\int_{-1+\beta b}^{\varepsilon\zeta} u^*_{\mathrm{sh}} v^*_{\mathrm{sh}} \right) \\
&+ \varepsilon\mu \overline{T u^*_{\mathrm{sh}}} \partial_x \overline{v} + \varepsilon\mu \frac{h^2}{3} \partial_x^2 \overline{u} \partial_x \mathrm{Q}_y \\
&+ \varepsilon\mu \frac{1}{2h} \partial_x \left(\int_{-1+\beta b}^{\varepsilon\zeta} v^*_{\mathrm{sh}} \left([1 + z - \beta b]^2 - \frac{h^2}{3} \right) \partial_x^2 \overline{u} \right) \\
&+ \varepsilon \max\left(\mu^{\frac{3}{2}}, \beta\sqrt{\mu}\right) R.
\end{aligned}
$$

Finally, we can compute that

$$\frac{1}{2}\int_{-1+\beta b}^{\varepsilon\zeta} v_{\rm sh}^{*}\left([1+z-\beta b]^{2}-\frac{h^{2}}{3}\right)=\frac{1}{24}h^{3}v^{\sharp},$$

and the first equation follows from Lemma 2.7. The second equation follows similarly using the fact that $v_{\rm sh}^{*}=v_{\rm sh}-\mathrm{Q}_{y}$. $\square$

As noticed in [4], the quantity E defined by

$$E=\begin{pmatrix}E_{xx} & E_{xy}\\ E_{xy} & E_{yy}\end{pmatrix}=\int_{-1+\beta b}^{\varepsilon\zeta}\mathbf{V}_{\rm sh}^{*}\otimes\mathbf{V}_{\rm sh}^{*} \tag{56}$$

appears in the equations of Q_{x} and Q_{y} and can not be express with respect to ζ, $\overline{\mathbf{V}}$ and $\mathbf{V}^{\sharp}$. The following subsection is devoting to giving an equation for E.

4.2 Equations for E

In this part, we derive an equation for E up to terms of order $\mathcal{O}(\mu)$. We have to introduce the quantity F

$$F=(F_{ijk})_{i,j,k}=\int_{-1+\beta b}^{\varepsilon\zeta}\mathbf{V}_{\rm sh}^{*}\otimes\mathbf{V}_{\rm sh}^{*}\otimes\mathbf{V}_{\rm sh}^{*}. \tag{57}$$

The following proposition gives an equation for E.

Proposition 4.3. *If* $\left(\zeta,\mathbf{U}_{/\!/}^{\mu,0},\boldsymbol{\omega}\right)$ *satisfy the Castro-Lannes system* (14)*, then E satisfies the following equation*

$$\partial_{t}E+\varepsilon\overline{u}\partial_{x}E+\varepsilon l\left(E,\partial_{x}\overline{\mathbf{V}}\right)+\varepsilon\sqrt{\mu}\partial_{x}F_{\cdot,\cdot,1}+\frac{\varepsilon}{\mathrm{Ro}}E^{S}=\left(\varepsilon\sqrt{\mu}\partial_{x}\overline{v}+\frac{\varepsilon\sqrt{\mu}}{\mathrm{Ro}}\right)\mathcal{D}(\mathbf{V}^{\sharp},\overline{u})$$
$$+\max\left(\varepsilon\mu,\varepsilon\beta\sqrt{\mu},\frac{\varepsilon}{\mathrm{Ro}}\mu\right)R,$$

where

$$E^{S}=\int_{-1+\beta b}^{\varepsilon\zeta}\mathbf{V}_{\rm sh}^{\perp}\otimes\mathbf{V}_{\rm sh}+\mathbf{V}_{\rm sh}\otimes\mathbf{V}_{\rm sh}^{\perp}=\begin{pmatrix}-2E_{xy} & E_{xx}-E_{yy}\\ E_{xx}-E_{yy} & 2E_{xy}\end{pmatrix} \tag{58}$$

$$l\left(E,\partial_{x}\overline{\mathbf{V}}\right)=\begin{pmatrix}3\partial_{x}\overline{u}E_{xx}+2\partial_{x}\overline{v}E_{xy} & 2\partial_{x}\overline{u}E_{xy}+\partial_{x}\overline{v}E_{yy}\\ 2\partial_{x}\overline{u}E_{xy}+\partial_{x}\overline{v}E_{yy} & \partial_{x}\overline{u}E_{yy}\end{pmatrix} \tag{59}$$

and

$$\mathcal{D}(\mathbf{V}^{\sharp},\overline{u})=\partial_{x}^{2}\overline{u}\begin{pmatrix}0 & u^{\sharp}\\ u^{\sharp} & 2v^{\sharp}\end{pmatrix}. \tag{60}$$

Proof. The proof is similar to the computation in Part 4.5.2 and Part 5.4.1 in [4]. We compute $\partial_t E$ and we use the second equations of Propositions 4.1 and 4.2 up to terms of order $\mathcal{O}(\mu)$. For the Coriolis contribution, we use the expansion of u and v given in Proposition 2.5 and 2.4. □

The quantity F appears in the equation of E and can not be expressed with respect to ζ, $\overline{\mathbf{V}}$, $\mathbf{V}^\sharp$ and E. The next proposition gives an equation for F up to terms of order $\mathcal{O}(\sqrt{\mu})$.

Proposition 4.4. *If* $\left(\zeta, \mathbf{U}_{/\!/}^{\mu,0}, \boldsymbol{\omega}\right)$ *satisfy the Castro-Lannes system* (14)*, then* F_{ijk} *satisfies the following equation*

$$\partial_t F_{ijk} + \varepsilon(\overline{u}\partial_x F_{ijk} + \partial_x \overline{u} F_{ijk} + F_{1kj}\partial_x \mathbf{V}_i + F_{i1k}\partial_x \mathbf{V}_j + F_{ij1}\partial_x \mathbf{V}_k) + \frac{\varepsilon}{\mathrm{Ro}} F^S = \max\left(\varepsilon, \frac{\varepsilon}{\mathrm{Ro}}\right)\sqrt{\mu} R,$$

where

$$F^S = \int_{-1+\beta b}^{\varepsilon\zeta} \mathbf{V}_{\mathrm{sh}}^{\perp} \otimes \mathbf{V}_{\mathrm{sh}} \otimes \mathbf{V}_{\mathrm{sh}} + \mathbf{V}_{\mathrm{sh}} \otimes \mathbf{V}_{\mathrm{sh}}^{\perp} \otimes \mathbf{V}_{\mathrm{sh}} + \mathbf{V}_{\mathrm{sh}} \otimes \mathbf{V}_{\mathrm{sh}} \otimes \mathbf{V}_{\mathrm{sh}}^{\perp}. \tag{61}$$

Proof. The proof is similar to the computation in Part 4.5.3 and Part 5.4.2 in [4]. We compute $\partial_t F$ and we use the second equations of Propositions 4.1 and 4.2 up to terms of order $\mathcal{O}(\sqrt{\mu})$. For the Coriolis contribution, we use the expansion of u and v in Proposition 2.5 and 2.4. □

4.3 The Green-Naghdi equations

We can now establish the Green-Naghdi equations when $d = 1$. The Green-Naghdi equations are the following system

$$\begin{cases} \partial_t \zeta + \partial_x (h\overline{u}) = 0, \\ (1+\mu\mathcal{T})(\partial_t \overline{u} + \varepsilon\overline{u}\partial_x\overline{u}) + \partial_x\zeta - \frac{\varepsilon}{\mathrm{Ro}}\overline{v} + \varepsilon\mu\mathcal{Q}(\overline{u}) + \varepsilon\mu\partial_x E_{xx} + \varepsilon\mu^{\frac{3}{2}}\mathcal{C}_1\big(u^\sharp, \overline{u}\big) + \frac{\varepsilon}{\mathrm{Ro}}\frac{\mu^{\frac{3}{2}}}{24h}\partial_x^2(h^3 v^\sharp) = 0, \\ \partial_t \overline{v} + \varepsilon\overline{u}\partial_x\overline{v} + \frac{\varepsilon}{\mathrm{Ro}}\overline{u} + \varepsilon\mu\partial_x E_{xy} + \varepsilon\mu^{\frac{3}{2}}\mathcal{C}_2\left(v^\sharp, \partial_x^2\overline{u}\right) = 0, \\ \partial_t \mathbf{V}^\sharp + \varepsilon\mathbf{V}^\sharp\partial_x\overline{u} + \varepsilon\overline{u}\partial_x\mathbf{V}^\sharp + \frac{\varepsilon}{\mathrm{Ro}}\mathbf{V}^{\sharp\perp} = 0, \\ \partial_t E + \varepsilon\overline{u}\partial_x E + \varepsilon\, l\big(E, \partial_x\overline{\mathbf{V}}\big) + \varepsilon\sqrt{\mu}\partial_x F_{\cdot,\cdot,1} + \frac{\varepsilon}{\mathrm{Ro}}E^S = \left(\varepsilon\sqrt{\mu}\partial_x\overline{v} + \frac{\varepsilon}{\mathrm{Ro}}\sqrt{\mu}\right)\mathcal{D}(\mathbf{V}^\sharp, \overline{u}), \\ \partial_t F_{ijk} + \varepsilon\overline{u}\partial_x F_{ijk} + \varepsilon\partial_x\overline{u}F_{ijk} + \varepsilon F_{1kj}\partial_x\mathbf{V}_i + \varepsilon F_{i1k}\partial_x\mathbf{V}_j + \varepsilon F_{ij1}\partial_x\mathbf{V}_k + \frac{\varepsilon}{\mathrm{Ro}}F^S = 0. \end{cases} \tag{62}$$

where

$$
\begin{aligned}
&\mathcal{T} = -\frac{1}{3h}\partial_x\left(h^3\partial_x\cdot\right),\\
&\mathcal{Q}(\overline{u}) = \frac{2}{3h}\partial_x\left(h^3\left[\partial_x\overline{u}\right]^2\right),\\
&\mathcal{C}_1\left(u^\sharp,\overline{u}\right) = -\frac{1}{6h}\partial_x\left(2h^3u^\sharp\partial_x^2\overline{u} + \partial_x(h^3u^\sharp)\partial_x\overline{u}\right),\\
&\mathcal{C}_2\left(v^\sharp,w\right) = -\frac{1}{24h}\partial_x\left(h^3v^\sharp w\right),\\
&l\left(E,\partial_x\overline{\mathbf{V}}\right) = \begin{pmatrix} 3\partial_x\overline{u}E_{xx} + 2\partial_x\overline{v}E_{xy} & 2\partial_x\overline{u}E_{xy} + \partial_x\overline{v}E_{yy} \\ 2\partial_x\overline{u}E_{xy} + \partial_x\overline{v}E_{yy} & \partial_x\overline{u}E_{yy}\end{pmatrix},\\
&\mathcal{D}(\mathbf{V}^\sharp,\overline{u}) = \partial_x^2\overline{u}\begin{pmatrix} 0 & u^\sharp \\ u^\sharp & 2v^\sharp\end{pmatrix}
\end{aligned}
\tag{63}
$$

and

$$
\begin{aligned}
&E^S = \int_{-1+\beta b}^{\varepsilon\zeta} \mathbf{V}_{\text{sh}}^{\perp}\otimes\mathbf{V}_{\text{sh}} + \mathbf{V}_{\text{sh}}\otimes\mathbf{V}_{\text{sh}}^{\perp} = \begin{pmatrix} -2E_{xy} & E_{xx}-E_{yy} \\ E_{xx}-E_{yy} & 2E_{xy}\end{pmatrix},\\
&F^S = \int_{-1+\beta b}^{\varepsilon\zeta} \mathbf{V}_{\text{sh}}^{\perp}\otimes\mathbf{V}_{\text{sh}}\otimes\mathbf{V}_{\text{sh}} + \mathbf{V}_{\text{sh}}\otimes\mathbf{V}_{\text{sh}}^{\perp}\otimes\mathbf{V}_{\text{sh}} + \mathbf{V}_{\text{sh}}\otimes\mathbf{V}_{\text{sh}}\otimes\mathbf{V}_{\text{sh}}^{\perp},
\end{aligned}
\tag{64}
$$

and $\mathbf{V}^\sharp$ is defined in (29), E in (56) and F in (57). Notice that the first, the second and the third equations of System (62) are the classical Green-Naghdi equations with new terms due to the vorticity (terms with $\mathbf{V}^\sharp$ and E). The last equations are important to get a close system. We can now state that the Green-Naghdi equations are an order $\mathcal{O}(\mu^2)$ approximation of the water waves equations.

Proposition 4.5. *In the Green-Naghdi regime with small topography variations $\mathcal{A}_{GN}$, the Castro-Lannes equations* (14) *are consistent at order $\mathcal{O}(\mu^2)$ with the Green-Naghdi equations* (62) *in the sense of Definition 1.4.*

Proof. The proof is similar to the one in Proposition 2.12. The first equation of the Green-Naghdi equations is always satisfied for a solution of the Castro-Lannes formulation by Proposition 2.3. For the second equation, we use Proposition 2.5, Proposition 4.1 together with Proposition 2.6, Lemma 2.7 and Proposition 2.10. Notice the fact that all the terms with Q_x disappear. The third equation follows from Proposition 2.4, 2.5 and 4.2 (all the terms with Q_y also disappear). The last equations follows from Propositions 2.10, 4.3 and 4.4. □

Remark 4.6. *Notice that even without a Coriolis forcing, we can not decrease the number of equations in the previous Green-Naghdi equations. However, if one also suppose that the vorticity is initially of the form $(0,\omega_y,0)^t$, which corresponds to the propagation of 2D water waves, we can significantly simplify the Green-Naghdi equations (See Section 4 in [4] and [18]).*

4.4 A simplified model in the case of a weak rotation and medium amplitude waves

As noticed in [4], if we assume that $\varepsilon = \mathcal{O}(\sqrt{\mu})$ we can simplify the Green-Naghdi equations. This regime corresponds to medium amplitude waves (in the terminology of [17]). We also assume that $\frac{\varepsilon}{\mathrm{Ro}} = \mathcal{O}(\sqrt{\mu})$. Then, we can simplify the Green-Naghdi system (62) by dropping all the terms of $\mathcal{O}(\mu^2)$ and we get

$$\begin{cases} \partial_t \zeta + \partial_x \left(h\overline{u}\right) = 0, \\ \left(1+\mu\mathcal{T}\right)\left(\partial_t \overline{u} + \varepsilon \overline{u}\partial_x \overline{u}\right) + \partial_x \zeta - \dfrac{\varepsilon}{\mathrm{Ro}}\overline{v} + \varepsilon\mu\mathcal{Q}(\overline{u}) + \varepsilon\mu\partial_x E_{xx} = 0, \\ \partial_t \overline{v} + \varepsilon\overline{u}\partial_x \overline{v} + \dfrac{\varepsilon}{\mathrm{Ro}}\overline{u} + \varepsilon\mu\partial_x E_{xy} = 0, \\ \partial_t E + \varepsilon\overline{u}\partial_x E + \varepsilon\, l\big(E, \partial_x \overline{\mathbf{V}}\big) + \dfrac{\varepsilon}{\mathrm{Ro}} E^S = 0. \end{cases} \tag{65}$$

Notice that in this regime, we catch effects of the vorticity on $\overline{\mathbf{V}}$ thanks to the second order tensor E. Without vorticity, this regime is particularly interesting since it is related to the Camassa-Holm equation and the Degasperis-Procesi equation (see for instance [6]). It could be interesting to understand how we can adapt these two scalar equations in presence of a Coriolis forcing.

Acknowledgments

The author has been partially funded by the ANR project Dyficolti ANR-13-BS01-0003.

References

[1] B. Alvarez-Samaniego and D. Lannes. Large time existence for 3D water-waves and asymptotics. *Invent. Math.*, 171(3):485–541, 2008.

[2] W. Ben Youssef and D. Lannes. The long wave limit for a general class of 2d quasilinear hyperbolic problems. *Communications in Partial Differential Equations*, 27(5-6):979–1020, 2002.

[3] J. L. Bona, T. Colin, and D. Lannes. Long wave approximations for water waves. *Arch. Ration. Mech. Anal.*, 178(3):373–410, 2005.

[4] A. Castro and D. Lannes. Fully nonlinear long-wave models in the presence of vorticity. *Journal of Fluid Mechanics*, 759:642–675, 11 2014.

[5] A. Castro and D. Lannes. Well-posedness and shallow-water stability for a new Hamiltonian formulation of the water waves equations with vorticity. *Indiana Univ. Math. J.*, 64(4):1169–1270, 2015.

[6] A. Constantin and D. Lannes. The hydrodynamical relevance of the camassa–holm and degasperis–procesi equations. *Archive for Rational Mechanics and Analysis*, 192(1):165–186, 2008.

[7] W. Craig. An existence theory for water waves and the Boussinesq and Korteweg-de Vries scaling limits. *Comm. Partial Differential Equations*, 10(8):787–1003, 1985.

[8] W. Craig, C. Sulem, and P.-L. Sulem. Nonlinear modulation of gravity waves: a rigorous approach. *Nonlinearity*, 5(2):497–522, 1992.

[9] V.N. Galkin and Yu.A. Stepanyants. On the existence of stationary solitary waves in a rotating fluid. *Journal of Applied Mathematics and Mechanics*, 55(6):939 – 943, 1991.

[10] J. P. Germain and D. P. Renouard. On permanent nonlinear waves in a rotating fluid. *Fluid Dynamics Research*, 7(5-6):263, 1991.

[11] A.E. Gill. *Atmosphere-Ocean Dynamics*. International Geophysics. Elsevier Science, 1982.

[12] R.H.J. Grimshaw, L.A. Ostrovsky, V.I. Shrira, and Yu. A. Stepanyants. Long nonlinear surface and internal gravity waves in a rotating ocean. *Surveys in Geophysics*, 19(4):289–338, 1998.

[13] L. Hörmander. *Lectures on nonlinear hyperbolic differential equations*, volume 26 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer-Verlag, Berlin, 1997.

[14] S. Israwi. Large time existence for 1D Green-Naghdi equations. *Nonlinear Anal.*, 74(1):81–93, 2011.

[15] T. Kano and T. Nishida. A mathematical justification for Korteweg-de Vries equation and Boussinesq equation of water surface waves. *Osaka J. Math.*, 23(2):389–413, 1986.

[16] C. E. Kenig, G. Ponce, and L. Vega. Well-posedness and scattering results for the generalized korteweg-de vries equation via the contraction principle. *Communications on Pure and Applied Mathematics*, 46(4):527–620, 1993.

[17] D. Lannes. *The water waves problem*, volume 188 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2013. Mathematical analysis and asymptotics.

[18] D. Lannes and F. Marche. Nonlinear wave-current interactions in shallow water. *Studies in Applied Mathematics*, 2016.

[19] D. Lannes and J. C. Saut. Weakly transverse Boussinesq systems and the Kadomtsev-Petviashvili approximation. *Nonlinearity*, 19(12):2853–2875, 2006.

[20] P.H. LeBlond and L.A. Mysak. *Waves in the Ocean*. Elsevier Oceanography Series. Elsevier Science, 1981.

[21] A. I. Leonov. The effect of the earth's rotation on the propagation of weak nonlinear surface and internal long oceanic waves. *Annals of the New York Academy of Sciences*, 373(1):150–159, 1981.

[22] Y. Li, J. Huang, and W. Yan. The cauchy problem for the ostrovsky equation with negative dispersion at the critical regularity. *Journal of Differential Equations*, 259(4):1379 – 1408, 2015.

[23] F. Linares and A. Milanés. Local and global well-posedness for the ostrovsky equation. *Journal of Differential Equations*, 222(2):325 – 340, 2006.

[24] A. Majda. *Introduction to PDEs and waves for the atmosphere and ocean*, volume 9 of *Courant Lecture Notes in Mathematics*. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003.

[25] A. I. Markovskii. Remarks on the lp-lq estimates of solutions of the klein-gordon equation. *Ukrainian Mathematical Journal*, 44(2):205–214, 1992.

[26] B. Mélinand. Coriolis effect on water waves. *arXiv:1511.07407*, 2015.

[27] L. Ostrovsky. Nonlinear internal waves in a rotating ocean. *Oceanology*, 18(2):119–125, 1978.

[28] L. A. Ostrovsky and Yu. A. Stepanyants. *Nonlinear Waves 3: Physics and Astrophysics Proceedings of the Gorky School 1989*, chapter Nonlinear Surface and Internal Waves in Rotating Fluids, pages 106–128. Springer Berlin Heidelberg, Berlin, Heidelberg, 1990.

[29] J. Pedlosky. *Geophysical Fluid Dynamics*. Springer study edition. Springer New York, 1992.

[30] G. L. Richard and S. L. Gavrilyuk. A new model of roll waves: comparison with Brock's experiments. *J. Fluid Mech.*, 698:374–405, 2012.

[31] G. Schneider and C. E. Wayne. Corrigendum: The long-wave limit for the water wave problem I. The case of zero surface tension [mr1780702]. *Comm. Pure Appl. Math.*, 65(5):587–591, 2012.

[32] F. J. Seabra-Santos, D. P. Renouard, and A. M. Temperville. Numerical and experimental study of the transformation of a solitary wave over a shelf or isolated obstacle. *Journal of Fluid Mechanics*, 176:117–134, 3 1987.

[33] H. Sverdrup. Dynamics of tides on the north siberian shelf. *Geophys. Publ.*, 4:1 – 75, 1927.

[34] V. M. Teshukov. Gas-dynamic analogy for vortex free-boundary flows. *Journal of Applied Mechanics and Technical Physics*, 48(3):303–309, 2007.

[35] V. Varlamov and Y. Liu. Cauchy problem for the ostrovsky equation. *Discrete and Continuous Dynamical Systems*, 10(3):731–753, 2004.

[36] W. Wahl. Lp-decacy rates for homogeneous wave-equations. *Mathematische Zeitschrift*, 120:93–106, 1971.

[37] V.E Zakharov. Stability of periodic waves of finite amplitude on the surface of a deep fluid. *J. Applied Mech. Tech. Phys.*, 9:190–194, 1968.

[38] P. Zhang and Y. Liu. Symmetry and uniqueness of the solitary-wave solution for the ostrovsky equation. *Archive for Rational Mechanics and Analysis*, 196(3):811–837, 6 2010.